Algebraic Solution of the Hubbard Model on the Infinite Interval

Shuichi MURAKAMI [1] ‡ and Frank GÖHMANN [2] § ¶

[1] *Department of Applied Physics, Faculty of Engineering,*
University of Tokyo,
Hongo 7-3-1, Bunkyo-ku, Tokyo 113, Japan.

[2] *Physikalisches Institut der Universität Bayreuth, TP1,*
95440 Bayreuth, Germany.

(Received:

Abstract

We develop the quantum inverse scattering method for the one-dimensional Hubbard model on the infinite line at zero density. This enables us to diagonalize the Hamiltonian algebraically. The eigenstates can be classified as scattering states of particles, bound pairs of particles and bound states of pairs. We obtain the corresponding creation and annihilation operators and calculate the S-matrix. The Hamiltonian on the infinite line is invariant under the Yangian quantum group Y(su(2)). We show that the n-particle scattering states transform like n-fold tensor products of fundamental representations of Y(su(2)) and that the bound states are Yangian singlet.

‡E-mail:`murakami@appi.t.u-tokyo.ac.jp`

§E-mail:`Frank.Goehmann@theo.phy.uni-bayreuth.de`

¶Address from September 1, 1997: Institute for Theoretical Physics, State University of New York at Stony Brook, Stony Brook, NY 11794-3840, USA

1 Introduction

The past two decades have seen a rapid development of algebraic methods for the exact solution of one-dimensional quantum systems. This development was partly initiated and most strongly influenced by the contribution of the Leningrad school, which clarified the fundamental meaning of the Yang-Baxter equation for the understanding of exactly solvable one-dimensional systems [1, 2, 3, 4]. It culminated in the invention of quantum groups [5] which are by now generally accepted as the mathematical framework of the theory.

The more traditional (coordinate) Bethe ansatz approach to one-dimensional quantum systems consists of a direct construction of eigenfunctions and yields a system of Bethe ansatz equations for a set of parameters which characterize these eigenfunctions and the spectrum of the Hamiltonian. One may ask, if the quantum inverse scattering method is more than an alternative way to derive the Bethe ansatz equations. In fact, as long as we are only interested in quantities, which are entirely determined by the spectrum of the Hamiltonian, we do not need the quantum inverse scattering method. The most successful attempts on the calculation of correlation functions [6, 7], however, rely heavily on the use of algebraic methods.

Unfortunately, some of the physically more interesting models, like the Hubbard model [8], the Kondo model [9, 10] or the Anderson model [9] have only partly been capable by an algebraic treatment so far. In the present article we will report on some recent progress concerning the Hubbard model.

The Hubbard model was solved by (coordinate) Bethe ansatz by Lieb and Wu [11, 12, 13]. The basic tools of the quantum inverse scattering method for the system under periodic boundary conditions, R-matrix and monodromy matrix, were constructed by Shastry [14, 15, 16] and by Olmedilla et al. [17, 18, 19]. Since there exists a so-called pseudo vacuum state, on which the monodromy matrix is acting tridiagonally and which is an eigenstate of its diagonal elements, an algebraic construction of eigenstates, usually called algebraic Bethe ansatz, should be possible according to common belief. However, algebraic Bethe ansatz, if we understand it in this broad sense, is not really a method. It merely means to use commutation relations between the matrix elements of the monodromy matrix, whose interpretation in physical terms is moreover not a priori clear, to construct the eigenstates of some suitably chosen generating function of a family of commuting operators, which contains the Hamilto-

nian of the system. The most sophisticated variant of algebraic Bethe ansatz was developed by Tarasov [20], who diagonalized the transfer matrix of the Izergin-Korepin model [21]. This work was recently generalized to the case of the Hubbard model in a remarkable article of Ramos and Martins [22]. Besides the fact that their proof of the "cancellation of unwanted terms" seems to be incomplete, we still feel unsatisfied about two points. First, the expressions for the eigenstates are of complicated recursive nature. It seems to be unlikely that they can be used in the calculation of correlation functions. Second, due to the complicated way in which the various operators contained in the monodromy matrix enter the expression for the eigenstates, an intuitive physical interpretation is difficult.

Therefore we follow a different route here [23] which is based on the work of one of us on the fermionic nonlinear Schrödinger model [24, 25]. We take the thermodynamic limit first and construct the eigenstates afterwards. This is the original idea of the quantum inverse scattering method [26, 27, 28], which was designed in close analogy to the inverse scattering theory for the solution of classical integrable systems [29]. The disadvantage of this method is that it is so far restricted to uncorrelated vacua (ground states). It is therefore neither capable of relativistic models nor of non-relativistic models with general nonzero density of particles in the ground state. On the other hand, there are lots of advantages which make it highly desirable to generalize the method. We get rid of the complicated Bethe ansatz equations. The algebra of the elements of the monodromy matrix simplifies, and we obtain an intuitive interpretation of these operators. Furthermore, there is a realistic perspective to calculate correlation functions by use of the quantum Gelfand-Levitan equation [30].

In the following section we summarize several results for the finite periodic system [17, 31], which will be needed later. In section 3 we describe the passage to the infinite interval. Some of the technical details are shifted to Appendix A. We obtain a renormalized monodromy matrix and the commutation relations among its entries, which are encoded in a new simplified R-matrix. In section 4 we identify a suitable generating function of commuting operators, namely the quantum determinant of a submatrix $A(\lambda)$ of the monodromy matrix. The commutation relations between the entries of this submatrix decouple from the rest of the algebra and can be written in form of an exchange relation which is generated by a submatrix $r(\lambda, \mu)$ of the new R-matrix. $r(\lambda, \mu)$ is 4×4 and after an appropriate reparametrization turns into the rational R-matrix of the isotropic Heisenberg spin chain. Thus $A(\lambda)$ provides a representation of the Yangian quantum group Y(su(2)). In section 5 we identify some of the elements

of the monodromy matrix as creation and annihilation operators of eigenstates of the Hubbard Hamiltonian. We discuss the structure of higher conserved quantities, and we calculate the action of the Yangian on the eigenstates. In the first part of section 6 we propose two pairs of normalized creation and annihilation operators of scattering states. These operators constitute representations of the right and left Zamolodchikov-Faddeev algebra, respectively, and mutually anticommute. We interpret them as generators of fermionic quasi-particles. The Zamolodchikov-Faddeev algebra provides their S-matrix. These operators create the known Bethe ansatz states on the infinite interval. We work out the action of the Yangian on the Bethe ansatz states. The Yangian mixes spin multiplets, which are degenerate in the thermodynamic limit. In the second part of section 6 we propose creation and annihilation operators of bound states of quasi-particles. These bound states correspond to the string states of coordinate Bethe ansatz [32]. We calculate their S-matrix. All of them are Yangian singlet. Section 7 is left for a summary and outlook. Appendix B contains a list of all commutation relations between the elements of the monodromy matrix. Appendix C is devoted to a discussion of the construction of bound state operators. In Appendix D we give explicit expressions for some higher conserved operators, which are needed for the discussion in section 5.

2 Hamiltonian and Monodromy Matrix under Periodic Boundary Conditions

The Hubbard model in its most basic single-band version is describing the dynamics of interacting electrons inside the conduction band of a solid. Its Hamiltonian is usually formulated in terms of creation and annihilation operators $c_{j\sigma}^\dagger,\ c_{j\sigma}$ of electrons in Wannier states,

$$\hat{H} = -\sum_{j,\sigma=\uparrow,\downarrow} (c_{j+1,\sigma}^\dagger c_{j,\sigma} + c_{j,\sigma}^\dagger c_{j+1,\sigma}) + U \sum_j \left[\left(n_{j\uparrow} - \frac{1}{2} \right) \left(n_{j\downarrow} - \frac{1}{2} \right) - \frac{1}{4} \right]. \tag{2.1}$$

The index j runs over all Wannier states which may be identified with the lattice sites of the solid. $\sigma =\uparrow, \downarrow$ is the spin index and $n_{j\sigma} = c_{j\sigma}^\dagger c_{j\sigma}$ is the density operator. The interaction part of the Hamiltonian is thought to model the screened Coulomb interaction of the electrons. In the following we will use the terminology of quantum field theory, and we will call the empty band state $|0\rangle$ the zero density vacuum. Since we want to study finite excitations over the zero density vacuum $|0\rangle$ in the thermodynamic limit, we normalized the Hamiltonian such

that $\hat{H}|0\rangle = 0$.

The most distinctive feature of the one-dimensional model is that its Hamiltonian may be embedded into a family of mutually commuting operators, which is generated by a properly constructed transfer matrix. This feature is commonly referred to as quantum integrability. Below we will give a short account of the construction of the transfer matrix and its related monodromy matrix for the Hubbard model under periodic boundary conditions. We start from the local exchange relation [18]

$$\mathcal{R}(\lambda,\mu)[\mathcal{L}_j(\lambda)\otimes_{\mathrm{s}}\mathcal{L}_j(\mu)] = [\mathcal{L}_j(\mu)\otimes_{\mathrm{s}}\mathcal{L}_j(\lambda)]\mathcal{R}(\lambda,\mu). \tag{2.2}$$

The symbol $\otimes_{\mathrm{s}}$ in this equation denotes the Grassmann direct product

$$[A\otimes_{\mathrm{s}} B]_{\alpha\gamma,\beta\delta} = (-1)^{[P(\alpha)+P(\beta)]P(\gamma)}A_{\alpha\beta}B_{\gamma\delta} \tag{2.3}$$

with grading $P(1) = P(4) = 0,\ P(2) = P(3) = 1$. The R-matrix $\mathcal{R}$ is a c-number matrix which encodes the commutation rules of the matrix elements of the L-matrix $\mathcal{L}_j$. These matrix elements are operators acting on the j-th Wannier state. We adopt the expressions for the matrices $\mathcal{R}$ and $\mathcal{L}_j$ in terms of two parameterizing functions $\alpha(\lambda),\ \gamma(\lambda)$ from ref. [18]. For later convenience, however, we shift the arguments of $\alpha(\lambda)$ and $\gamma(\lambda)$ by $\frac{\pi}{4}$, such that we simply have $\alpha(\lambda) = \cos\lambda$, $\gamma(\lambda) = \sin\lambda$. Then the L-matrix is

$$\mathcal{L}_j(\lambda) = \begin{pmatrix} -\mathrm{e}^{h(\lambda)}f_{j\uparrow}f_{j\downarrow} & -f_{j\uparrow}c_{j\downarrow} & \mathrm{i}c_{j\uparrow}f_{j\downarrow} & \mathrm{i}c_{j\uparrow}c_{j\downarrow}\mathrm{e}^{h(\lambda)} \\ -\mathrm{i}f_{j\uparrow}c_{j\downarrow}^\dagger & \mathrm{e}^{-h(\lambda)}f_{j\uparrow}g_{j\downarrow} & \mathrm{e}^{-h(\lambda)}c_{j\uparrow}c_{j\downarrow}^\dagger & \mathrm{i}c_{j\uparrow}g_{j\downarrow} \\ c_{j\uparrow}^\dagger f_{j\downarrow} & \mathrm{e}^{-h(\lambda)}c_{j\uparrow}^\dagger c_{j\downarrow} & \mathrm{e}^{-h(\lambda)}g_{j\uparrow}f_{j\downarrow} & g_{j\uparrow}c_{j\downarrow} \\ -\mathrm{i}\mathrm{e}^{h(\lambda)}c_{j\uparrow}^\dagger c_{j\downarrow}^\dagger & c_{j\uparrow}^\dagger g_{j\downarrow} & \mathrm{i}g_{j\uparrow}c_{j\downarrow}^\dagger & -g_{j\uparrow}g_{j\downarrow}\mathrm{e}^{h(\lambda)} \end{pmatrix}, \tag{2.4}$$

where $f_{j\sigma}(\lambda) = (1-n_{j\sigma})\sin\lambda + \mathrm{i}n_{j\sigma}\cos\lambda$, $g_{j\sigma}(\lambda) = (1-n_{j\sigma})\cos\lambda - \mathrm{i}n_{j\sigma}\sin\lambda$, and $h(\lambda)$ is defined as

$$\frac{\sinh 2h(\lambda)}{\sin 2\lambda} = \frac{U}{4}. \tag{2.5}$$

Due to space limitations we do not reproduce the R-matrix here. It is 16×16 and contains 36 non-vanishing entries, only ten of which are different modulo signs. The ten different entries are denoted by ρ_i, $i = 1,\cdots,10$, in ref. [18]. They are rational functions of $\cos\lambda$, $\sin\lambda$ and $\mathrm{e}^{h(\lambda)}$. A list of the matrix elements and some basic formulae which have been used in our calculations can be found in Appendix A of ref. [23]. Eq.(2.2) considered as an abstract definition of an algebra has the property that tensor products of representation of this algebra

are again representations. This property is called co-multiplication property. It assures that the monodromy matrix

$$\mathcal{T}_{mn}(\lambda) = \mathcal{L}_{m-1}(\lambda)\mathcal{L}_{m-2}(\lambda)\cdots\mathcal{L}_n(\lambda) \quad (m > n) \tag{2.6}$$

satisfies again eq.(2.2). $\mathcal{T}_{mn}(\lambda)$ contains all information about the Hubbard model under periodic boundary conditions. After proper renormalization $\mathcal{T}_{mn}(\lambda)$ turns into the monodromy matrix of the Hubbard model in the thermodynamic limit, which will be the central object of investigation of the present paper. Before we continue with the description of the thermodynamic limit, we list the most important properties of $\mathcal{T}_{mn}(\lambda)$ [31].

The transfer matrix

$$\tau_{mn}(\lambda) = \mathrm{str}(\mathcal{T}_{mn}(\lambda)) = \mathrm{tr}((\sigma^z \otimes \sigma^z)\mathcal{T}_{mn}(\lambda)) \tag{2.7}$$

generates a family of mutually commuting operators [14, 15, 18, 31],

$$\begin{aligned} \ln \tau_{mn}(\lambda) \quad &= \quad \frac{\mathrm{i}\pi}{2}(\hat{N}_{mn} - 2(m-n)) + \mathrm{i}\hat{\Pi}_{mn} \\ &\quad + \lambda\left(\hat{H}_{mn} + (m-n)\frac{U}{4}\right) + O(\lambda^2). \end{aligned} \tag{2.8}$$

Here

$$\hat{N}_{mn} = \sum_{j=n}^{m-1}(n_{j\uparrow} + n_{j\downarrow}) \tag{2.9}$$

is the particle number operator, and $\hat{\Pi}_{mn}$ is the lattice momentum operator. For the subtle question how to define $\hat{\Pi}_{mn}$ properly such that it commutes with the Hamiltonian we refer the reader to Ref. [31]. $\hat{H}_{mn}$ is the Hamiltonian (2.1) on a one-dimensional lattice of length $m-n$ under periodic boundary conditions (let j run from n to $m-1$ in eq.(2.1) and let $c_m = c_n$).

Clearly, the Hamiltonian (2.1) is invariant under su(2)-rotations generated by the operators of total spin

$$S^a = \frac{1}{2}\sum_{j=n}^{m-1} \sigma^a_{\alpha\beta} c^\dagger_{j\alpha} c_{j\beta}. \tag{2.10}$$

Here $a = x, y, z$ and the matrices σ^a are the Pauli matrices. Due to the invariance of the Hamiltonian under the transformation

$$c_{j\uparrow} \to c_{j\uparrow}, \quad c_{j\downarrow} \to (-1)^j c^\dagger_{j\downarrow}, \quad U \to -U \tag{2.11}$$

there exists a second su(2) symmetry [33, 34, 35]. This symmetry is called η-pairing symmetry. Applying (2.11) to (2.10) we get its generators η^a in the form

$$\eta^x = \frac{1}{2}\sum_{j=n}^{m-1}(-1)^j(c_{j\uparrow}^\dagger c_{j\downarrow}^\dagger + c_{j\downarrow}c_{j\uparrow}), \tag{2.12}$$

$$\eta^y = -\frac{\mathrm{i}}{2}\sum_{j=n}^{m-1}(-1)^j(c_{j\uparrow}^\dagger c_{j\downarrow}^\dagger - c_{j\downarrow}c_{j\uparrow}), \tag{2.13}$$

$$\eta^z = \frac{1}{2}(\hat{N}_{mn} - m + n). \tag{2.14}$$

Note that the transformation (2.11) twists the boundary conditions, if $m-n$ is odd. Therefore $\hat{H}_{mn}$ commutes with η^x and η^y only if the lattice has an even number of sites. Since η^z is essentially the particle number operator, we may understand the η-pairing symmetry as a non-Abelian extension of gauge symmetry.

Rotational symmetry and η-pairing symmetry both extend to symmetries of the monodromy matrix [31]. In order to make this statement explicit, we have to introduce certain matrix representations of su(2). We will denote the $n \times n$ unit matrix by I_n. Let

$$\Sigma_s^x = \frac{1}{2}(\sigma^+ \otimes \sigma^- + \sigma^- \otimes \sigma^+), \tag{2.15}$$

$$\Sigma_s^y = -\frac{\mathrm{i}}{2}(\sigma^+ \otimes \sigma^- - \sigma^- \otimes \sigma^+), \tag{2.16}$$

$$\Sigma_s^x = \frac{1}{4}(\sigma^z \otimes I_2 - I_2 \otimes \sigma^z), \tag{2.17}$$

and

$$\Sigma_\eta^x = \frac{1}{2}(\sigma^+ \otimes \sigma^+ + \sigma^- \otimes \sigma^-), \tag{2.18}$$

$$\Sigma_\eta^y = -\frac{\mathrm{i}}{2}(\sigma^+ \otimes \sigma^+ - \sigma^- \otimes \sigma^-), \tag{2.19}$$

$$\Sigma_\eta^z = \frac{1}{4}(\sigma^z \otimes I_2 + I_2 \otimes \sigma^z). \tag{2.20}$$

These matrices obviously satisfy the su(2) commutation rules

$$[\Sigma_j^a, \Sigma_j^b] = \mathrm{i}\varepsilon^{abc}\Sigma_j^c, \ \ j = s, \eta. \tag{2.21}$$

Let us perform a basis transformation

$$\tilde{\Sigma}_j^a = G^{ab}\Sigma_j^b \tag{2.22}$$

with transformation matrix

$$G = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.23}$$

Since G is orthogonal with $\det(G) = 1$, the transformed matrices $\tilde{\Sigma}_j^a$ satisfy (2.21). We are now able to state the su(2)⊕su(2) symmetry of the monodromy matrix $\mathcal{T}_{mn}(\lambda)$,

$$[\mathcal{T}_{mn}(\lambda), \tilde{\Sigma}_s^a + S^a] = 0, \tag{2.24}$$

$$[\mathcal{T}_{mn}(\lambda), \tilde{\Sigma}_\eta^a + \eta^a] = 0. \tag{2.25}$$

If $a = x, y$ in (2.25) we have to require both m and n to be odd. (2.24) and (2.25) imply the invariance of all higher conserved quantities in the expansion (2.8) under rotations and η-pairing transformations.

The twist (2.22) may appear somewhat unnatural. However, we had to introduce it here, since we wanted to keep the notation of the earlier paper [23]. We may remove the twist by a gauge transformation in auxiliary space. Let

$$W = \mathrm{diag}(1, 1, \mathrm{i}, \mathrm{i}). \tag{2.26}$$

Then

$$\tilde{\Sigma}_j^a = W^{-1} \Sigma_j^a W. \tag{2.27}$$

Thus the gauge transformed monodromy matrix $W \mathcal{T}_{mn}(\lambda) W^{-1}$ satisfies (2.24) and (2.25) with $\tilde{\Sigma}_j^a$ replaced by Σ_j^a. Note that $W \otimes W$ commutes with the R-matrix [31], which implies that the exchange relations for $\mathcal{T}_{mn}(\lambda)$ and $W \mathcal{T}_{mn}(\lambda) W^{-1}$ are the same.

The grading of the monodromy matrix, its behavior under particle-hole transformations [31] and the structure of the matrices Σ_j^a suggest the following block notation for the monodromy matrix,

$$\mathcal{T}_{mn}(\lambda) = \begin{pmatrix} D_{11}(\lambda) & C_{11}(\lambda) & C_{12}(\lambda) & D_{12}(\lambda) \\ B_{11}(\lambda) & A_{11}(\lambda) & A_{12}(\lambda) & B_{12}(\lambda) \\ B_{21}(\lambda) & A_{21}(\lambda) & A_{22}(\lambda) & B_{22}(\lambda) \\ D_{21}(\lambda) & C_{21}(\lambda) & C_{22}(\lambda) & D_{22}(\lambda) \end{pmatrix}. \tag{2.28}$$

We will see in the following that many of the algebraic properties of the Hubbard model are conveniently expressed in terms of the 2×2 submatrices $A(\lambda)$, $B(\lambda)$, $C(\lambda)$, $D(\lambda)$. As an

example let us describe the behavior of $\mathcal{T}_{mn}(\lambda)$ under hermitian conjugation, which will be needed later and which can be obtained by using the methods outlined in Ref. [31],

$$
\begin{aligned}
(A(\lambda))^\dagger &= \sigma^y A(\pi/2-\lambda^*)\sigma^y, & (2.29)\\
(B(\lambda))^\dagger &= \mathrm{i}\sigma^y B(\pi/2-\lambda^*)\sigma^y, & (2.30)\\
(C(\lambda))^\dagger &= \mathrm{i}\sigma^y C(\pi/2-\lambda^*)\sigma^y, & (2.31)\\
(D(\lambda))^\dagger &= \sigma^y D(\pi/2-\lambda^*)\sigma^y. & (2.32)
\end{aligned}
$$

The dagger on the lhs of these equations means hermitian conjugation in quantum space, and the asterisk on the rhs denotes complex conjugation. For notational convenience we did not attach labels m and n to the submatrices $A(\lambda), \cdots, D(\lambda)$ on the rhs of (2.28). We will keep the same notation below, when we discuss the thermodynamic limit.

3 Passage to the Infinite Interval

As we shall see in the sequel, carrying out the thermodynamic limit leads to a severe simplification of the R-matrix. The commutation relations between the entries of the monodromy matrix will become simple enough to allow for an identification of creation and annihilation operators of quasi-particles, generators of conserved quantities and symmetry operators. The thermodynamic limit cannot be taken naïvely. The monodromy matrix requires infrared renormalization, which has to be done with respect to a given vacuum characterized by macroscopic parameters. These are the density of electrons ρ_N and the magnetization density ρ_M. As result of the thermodynamic limit we will obtain the finite energy excitations over the chosen vacuum. In contrast to the algebraic Bethe ansatz for the finite periodic system we will not be able anymore to distinguish between a pseudo-vacuum, upon which all eigenstates of the transfer matrix are built by the action of creation operators, and the physical vacuum, which is the true ground state of the model. In general, in the thermodynamic limit both states will be characterized by different values of ρ_M, ρ_N and thus will be separated by an infinite energy difference.

Infrared renormalization of the monodromy matrix is done in analogy with the inverse scattering method for integrable classical systems [29] by splitting off the asymptotics of the vacuum expectation value of the monodromy matrix for $m, -n \to \infty$, which therefore has to be known a priori. For this reason the method [26, 27, 23, 28] is so far restricted to

(asymptotically) uncorrelated vacua. In case of the Hubbard model there are four possible choices, the empty band ($\rho_M = \rho_N = 0$), the completely filled band ($\rho_M = 0, \rho_N = 2$), and the half-filled band with all spins up ($\rho_M = 1, \rho_N = 1$) or all spins down ($\rho_M = -1, \rho_N = 1$). In the following we will restrict ourselves to the empty band vacuum $|0\rangle$ which is defined by

$$c_{j\sigma}|0\rangle = 0. \tag{3.1}$$

Our description of the general method closely follows Sklyanin [28]. Define the Hilbert space $\mathcal{H}$ of states of "compact support" as the space of all finite linear combinations of vectors $c^\dagger_{m_1\sigma_1} \cdots c^\dagger_{m_n\sigma_n}|0\rangle$. Denote the vacuum expectation value of the L-matrix by

$$V(\lambda) = \langle 0|\mathcal{L}_m(\lambda)|0\rangle. \tag{3.2}$$

$V(\lambda)$ does not depend on m, because of the translational invariance of the vacuum.

Let

$$\tilde{\mathcal{L}}_j(\lambda) = V(\lambda)^{-j-1}\mathcal{L}_j(\lambda)V(\lambda)^j, \tag{3.3}$$

$$\tilde{\mathcal{T}}_{mn}(\lambda) = V(\lambda)^{-m}\mathcal{T}_{mn}(\lambda)V(\lambda)^n. \tag{3.4}$$

It is easy to see that the limits $\lim_{n\to-\infty}\langle x|\tilde{\mathcal{T}}_{mn}(\lambda)|y\rangle$ and $\lim_{m\to\infty}\langle x|\tilde{\mathcal{T}}_{mn}(\lambda)|y\rangle$ exist for all $|x\rangle, |y\rangle \in \mathcal{H}$. These weak limits determine a pair of operators

$$\tilde{\mathcal{T}}^+_m(\lambda) = \lim_{n\to+\infty} \tilde{\mathcal{T}}_{nm}(\lambda), \tag{3.5}$$

$$\tilde{\mathcal{T}}^-_m(\lambda) = \lim_{n\to-\infty} \tilde{\mathcal{T}}_{mn}(\lambda), \tag{3.6}$$

with asymptotics

$$\lim_{m\to+\infty} \tilde{\mathcal{T}}^+_m(\lambda) = \lim_{m\to-\infty} \tilde{\mathcal{T}}^-_m(\lambda) = I_4. \tag{3.7}$$

Multiplying (2.6) from the left by $\mathcal{L}_m(\lambda)$ or from the right by $\mathcal{L}_{n-1}(\lambda)$, respectively, we obtain two recursion relations for $\mathcal{T}_{mn}(\lambda)$, which imply a pair of recursion relations for $\tilde{\mathcal{T}}^+_m(\lambda)$ and $\tilde{\mathcal{T}}^-_m(\lambda)$. By use of the asymptotic condition (3.7) these are equivalent to the following pair of Volterra "integral equations" for $\tilde{\mathcal{T}}^\pm_m(\lambda)$,

$$\tilde{\mathcal{T}}^+_m(\lambda) = I_4 + \sum_{j=m+1}^{\infty} \tilde{\mathcal{T}}^+_j(\lambda)\ (\tilde{\mathcal{L}}_{j-1}(\lambda) - I_4), \tag{3.8}$$

$$\tilde{\mathcal{T}}^-_m(\lambda) = I_4 + \sum_{j=-\infty}^{m-1} (\tilde{\mathcal{L}}_j(\lambda) - I_4)\ \tilde{\mathcal{T}}^-_j(\lambda). \tag{3.9}$$

The above considerations imply the existence of the weak limit

$$\tilde{\mathcal{T}}(\lambda) = \lim_{m,-n\to\infty} \tilde{\mathcal{T}}_{mn}(\lambda) = \tilde{\mathcal{T}}_m^+(\lambda)\tilde{\mathcal{T}}_m^-(\lambda). \tag{3.10}$$

$\tilde{\mathcal{T}}(\lambda)$ is the renormalized monodromy matrix. Equation (3.8), or (3.9) respectively, implies the "integral representation"

$$\tilde{\mathcal{T}}(\lambda) = I_4 + \sum_m (\tilde{\mathcal{L}}_m(\lambda) - I_4) + \sum_{m>n} (\tilde{\mathcal{L}}_m(\lambda) - I_4)(\tilde{\mathcal{L}}_n(\lambda) - I_4) + \cdots . \tag{3.11}$$

Note that $\langle 0|(\tilde{\mathcal{L}}_m(\lambda) - I_4)|0\rangle = 0$ by construction. Hence $\langle 0|\tilde{\mathcal{T}}(\lambda)|0\rangle = I_4$.

$\tilde{\mathcal{L}}_m(\lambda)$ can be easily calculated. We find

$$V(\lambda) = \mathrm{diag}(-\mathrm{e}^{h(\lambda)}\sin^2\lambda, \mathrm{e}^{-h(\lambda)}\cos\lambda\sin\lambda, \mathrm{e}^{-h(\lambda)}\cos\lambda\sin\lambda, -\mathrm{e}^{h(\lambda)}\cos^2\lambda), \tag{3.12}$$

and thus

$$\begin{aligned}
\tilde{\mathcal{L}}_m(\lambda) &= V(\lambda)^{-m-1}\mathcal{L}_m(\lambda)V(\lambda)^m \\
&= \begin{pmatrix}
(\mathrm{i}\cot\lambda)^{n_{m\uparrow}+n_{m\downarrow}} & (\mathrm{i}\cot\lambda)^{n_{m\uparrow}} c_{m\downarrow}\frac{\mathrm{e}^{-h(\lambda)}}{\sin\lambda}\mathrm{e}^{\mathrm{i}mp(\lambda)} & -\mathrm{i}c_{m\uparrow}(\mathrm{i}\cot\lambda)^{n_{m\downarrow}}\frac{\mathrm{e}^{-h(\lambda)}}{\sin\lambda}\mathrm{e}^{\mathrm{i}mp(\lambda)} & -\mathrm{i}c_{m\uparrow}c_{m\downarrow}\frac{1}{\sin^2\lambda}\cot^{2m}\lambda \\
-\mathrm{i}(\mathrm{i}\cot\lambda)^{n_{m\uparrow}} c_{m\downarrow}^\dagger \frac{\mathrm{e}^{h(\lambda)}}{\cos\lambda}\mathrm{e}^{-\mathrm{i}mp(\lambda)} & (\mathrm{i}\cot\lambda)^{n_{m\uparrow}-n_{m\downarrow}} & c_{m\uparrow}c_{m\downarrow}^\dagger\frac{1}{\sin\lambda\cos\lambda} & \mathrm{i}c_{m\uparrow}(\mathrm{i}\cot\lambda)^{-n_{m\downarrow}}\frac{\mathrm{e}^{h(\lambda)}}{\sin\lambda}\mathrm{e}^{\mathrm{i}mk(\lambda)} \\
c_{m\uparrow}^\dagger(\mathrm{i}\cot\lambda)^{n_{m\downarrow}}\frac{\mathrm{e}^{h(\lambda)}}{\cos\lambda}\mathrm{e}^{-\mathrm{i}mp(\lambda)} & c_{m\uparrow}^\dagger c_{m\downarrow}\frac{1}{\sin\lambda\cos\lambda} & (\mathrm{i}\cot\lambda)^{-n_{m\uparrow}+n_{m\downarrow}} & (\mathrm{i}\cot\lambda)^{-n_{m\uparrow}}c_{m\downarrow}\frac{\mathrm{e}^{h(\lambda)}}{\sin\lambda}\mathrm{e}^{\mathrm{i}mk(\lambda)} \\
\mathrm{i}c_{m\uparrow}^\dagger c_{m\downarrow}^\dagger\frac{1}{\cos^2\lambda}\tan^{2m}\lambda & -c_{m\uparrow}^\dagger(\mathrm{i}\cot\lambda)^{-n_{m\downarrow}}\frac{\mathrm{e}^{-h(\lambda)}}{\cos\lambda}\mathrm{e}^{-\mathrm{i}mk(\lambda)} & -\mathrm{i}(\mathrm{i}\cot\lambda)^{-n_{m\uparrow}}c_{m\downarrow}^\dagger\frac{\mathrm{e}^{-h(\lambda)}}{\cos\lambda}\mathrm{e}^{-\mathrm{i}mk(\lambda)} & (\mathrm{i}\cot\lambda)^{-n_{m\uparrow}-n_{m\downarrow}}
\end{pmatrix} .
\end{aligned} \tag{3.13}$$

Here we have introduced new functions

$$\mathrm{e}^{\mathrm{i}k(\lambda)} = -\mathrm{e}^{2h(\lambda)}\cot\lambda, \quad \mathrm{e}^{\mathrm{i}p(\lambda)} = -\mathrm{e}^{-2h(\lambda)}\cot\lambda, \tag{3.14}$$

which we adopted from the recent analytic Bethe Ansatz for the Hubbard model by Yue and Deguchi [36].

Now we will turn to the calculation of the commutation relations between the elements of $\tilde{\mathcal{T}}(\lambda)$. Let

$$\mathcal{L}_m^{(2)}(\lambda,\mu) = \mathcal{L}_m(\lambda) \otimes_{\mathrm{s}} \mathcal{L}_m(\mu), \tag{3.15}$$

$$\mathcal{T}_{mn}^{(2)}(\lambda,\mu) = \mathcal{T}_{mn}(\lambda) \otimes_{\mathrm{s}} \mathcal{T}_{mn}(\mu). \tag{3.16}$$

We may apply the above discussion of the renormalization of $\mathcal{T}_{mn}(\lambda)$ to $\mathcal{T}^{(2)}_{mn}(\lambda,\mu)$, if we replace $V(\lambda)$ by

$$V^{(2)}(\lambda,\mu) = \langle 0|\mathcal{L}_m(\lambda) \otimes_{\rm s} \mathcal{L}_m(\mu)|0\rangle. \tag{3.17}$$

Note that $V^{(2)}(\lambda,\mu)$ is not just the tensor product $V(\lambda)\otimes_{\rm s} V(\mu)$. There appear additional off-diagonal terms due to normal ordering of the operators. We obtain a renormalized tensor product matrix

$$\tilde{\mathcal{T}}^{(2)}(\lambda,\mu) = \lim_{m,-n\to\infty} V^{(2)}(\lambda,\mu)^{-m}\mathcal{T}^{(2)}_{mn}(\lambda,\mu)V^{(2)}(\lambda,\mu)^{n}, \tag{3.18}$$

which satisfies $\langle 0|\tilde{\mathcal{T}}^{(2)}(\lambda,\mu)|0\rangle = I_{16}$. Taking the vacuum expectation value of the local exchange relation (2.2) yields

$$\mathcal{R}(\lambda,\mu)V^{(2)}(\lambda,\mu) = V^{(2)}(\mu,\lambda)\mathcal{R}(\lambda,\mu), \tag{3.19}$$

and we conclude that

$$\mathcal{R}(\lambda,\mu)\tilde{\mathcal{T}}^{(2)}(\lambda,\mu) = \tilde{\mathcal{T}}^{(2)}(\mu,\lambda)\mathcal{R}(\lambda,\mu). \tag{3.20}$$

If $\tilde{\mathcal{T}}^{(2)}_{mn}(\lambda,\mu)$ is defined in analogy to $\tilde{\mathcal{T}}_{mn}(\lambda)$ with $V(\lambda)$ replaced by $V^{(2)}(\lambda,\mu)$ in the definition (3.4), then

$$\tilde{\mathcal{T}}_{mn}(\lambda)\otimes_{\rm s}\tilde{\mathcal{T}}_{mn}(\mu) = U_m(\lambda,\mu)^{-1}\tilde{\mathcal{T}}^{(2)}_{mn}(\lambda,\mu)U_n(\lambda,\mu), \tag{3.21}$$

where we have set

$$U_n(\lambda,\mu) = V^{(2)}(\lambda,\mu)^{-n}[V(\lambda)^n \otimes_{\rm s} V(\mu)^n]. \tag{3.22}$$

Assume for a while that the limits

$$U_+(\lambda,\mu)^{-1} = \lim_{m\to\infty} U_m(\lambda,\mu)^{-1}, \quad U_-(\lambda,\mu) = \lim_{m\to-\infty} U_m(\lambda,\mu). \tag{3.23}$$

exist in a common domain of convergence. Then, according to eq. (3.21) $\tilde{\mathcal{T}}_{mn}(\lambda)\otimes_{\rm s}\tilde{\mathcal{T}}_{mn}(\mu)$ has a weak limit for $m,-n\to\infty$. We identify this limit with $\tilde{\mathcal{T}}(\lambda)\otimes_{\rm s}\tilde{\mathcal{T}}(\mu)$,

$$\tilde{\mathcal{T}}(\lambda)\otimes_{\rm s}\tilde{\mathcal{T}}(\mu) = U_+(\lambda,\mu)^{-1}\tilde{\mathcal{T}}^{(2)}(\lambda,\mu)U_-(\lambda,\mu). \tag{3.24}$$

Finally, inserting the above equation into (3.20), we arrive at the exchange relation for the monodromy matrix $\tilde{\mathcal{T}}(\lambda)$ on the infinite interval,

$$\tilde{\mathcal{R}}^{(+)}(\lambda,\mu)\left[\tilde{\mathcal{T}}(\lambda)\otimes_{\rm s}\tilde{\mathcal{T}}(\mu)\right] = \left[\tilde{\mathcal{T}}(\mu)\otimes_{\rm s}\tilde{\mathcal{T}}(\lambda)\right]\tilde{\mathcal{R}}^{(-)}(\lambda,\mu), \tag{3.25}$$

where

$$\tilde{\mathcal{R}}^{(\pm)}(\lambda,\mu) = U_\pm(\mu,\lambda)^{-1}\mathcal{R}(\lambda,\mu)U_\pm(\lambda,\mu). \tag{3.26}$$

The calculation of the matrices $U_{\pm}(\lambda,\mu)$ is rather technical. We present it in Appendix A. Here we only note three important facts. (i) there is no common domain of convergence for all matrix elements of $U_{+}(\lambda,\mu)$ and $U_{-}(\lambda,\mu)$. We will come back to this point later. (ii) if we stay away from some singular points (cf. Appendix A) then $U_{+}(\lambda,\mu)_{\alpha\beta,\gamma\delta} = U_{-}(\lambda,\mu)_{\alpha\beta,\gamma\delta}$. (iii) it is a nontrivial matter of fact that all the matrix elements of $U_{\pm}(\lambda,\mu)$ are simple rational functions of the original Boltzmann weights $\rho_j(\lambda,\mu)$.

Using the explicit form of the matrices $U_{\pm}(\lambda,\mu)$ given in Appendix A, we obtain

$$\tilde{\mathcal{R}}(\lambda,\mu) = \tilde{\mathcal{R}}^{(+)}(\lambda,\mu) = \tilde{\mathcal{R}}^{(-)}(\lambda,\mu) =$$

$$\begin{pmatrix}
\rho_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{\rho_1\rho_4}{\mathrm{i}\rho_{10}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\rho_1\rho_4}{\mathrm{i}\rho_{10}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-\rho_1\rho_4}{\rho_5-\rho_4} & 0 & 0 & 0 \\
0 & -\mathrm{i}\rho_{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \rho_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{\rho_3\rho_4-\rho_2^2}{\rho_3-\rho_1} & 0 & 0 & \frac{\rho_9\rho_{10}}{\rho_3-\rho_1} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\mathrm{i}\rho_1\rho_4}{\rho_9} & 0 & 0 \\
0 & 0 & -\mathrm{i}\rho_{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{\rho_9\rho_{10}}{\rho_3-\rho_1} & 0 & 0 & \frac{\rho_3\rho_4-\rho_2^2}{\rho_3-\rho_1} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho_4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\mathrm{i}\rho_1\rho_4}{\rho_9} & 0 \\
0 & 0 & 0 & \rho_1-\rho_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{i}\rho_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{i}\rho_9 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho_1
\end{pmatrix} . \tag{3.27}$$

The reader is urged to compare this expression with the R-matrix on the finite interval [18]. Instead of the 36 non-vanishing elements of the original R-matrix we have only 18 non-vanishing elements here, which brings about simpler commutation relations between the elements of the monodromy matrix. All matrix elements except the two diagonal elements $\frac{\rho_3\rho_4-\rho_2^2}{\rho_3-\rho_1}$ are just at the position of the 1's of the permutation matrix, which means that, were it not for the two elements $\frac{\rho_3\rho_4-\rho_2^2}{\rho_3-\rho_1}$, all commutation relations would reduce to the mere interchange of two factors along with a multiplication by some rational function of the

Boltzmann weights. A close investigation of (3.25) shows in particular, that the elements of the submatrices $A(\lambda), \cdots, D(\lambda)$ generate sub-algebras of the exchange relation. The sub-algebras generated by $A(\lambda)$ and by $D(\lambda)$ are again Yang-Baxter algebras with certain new R-matrices which are submatrices of $\tilde{\mathcal{R}}(\lambda, \mu)$. The sub-algebras generated by $B(\lambda)$ and $C(\lambda)$ can, after certain normalizations, be identified as representations of the right and left Zamolodchikov-Faddeev algebra. The construction of these algebras and the discussion of their physical meaning will be the contents of the following sections.

Let us come back to the remark (i) above. Due to the peculiar convergence properties of the limits $U_\pm(\lambda, \mu)$ the exchange relation in the form (3.25) has only symbolical meaning. It has to be understood as generating a set of 256 equations, each of which is meaningful. These equations, however, are not necessarily defined on the same domain in the λ, μ parameter space (cf. Appendix A).

4 Yangian Symmetry and Commuting Operators

The definite goal of this work is to construct algebraically the eigenstates of the Hubbard Hamiltonian (2.1). As usual in the theory of integrable systems, we will not directly work with the Hamiltonian, but with an appropriately chosen generating function of a whole family of mutually commuting operators. For the finite periodic system this generating function is the logarithm of the transfer matrix $\tau_{mn}(\lambda)$. Commuting operators are obtained as the coefficients of its expansion around $\lambda = 0$ (2.8). Equations (3.11) and (3.13) show that this expansion does not exist for the renormalized monodromy matrix $\tilde{\mathcal{T}}(\lambda)$. There is substitute, however, which is intimately connected with the existence of an additional Y(su(2)) quantum group symmetry of the Hubbard model in the thermodynamic limit [23]. We shall describe it below.

The commutation relations between the elements of the submatrix $A(\lambda)$ decouple from the rest of the algebra.

$$r(\lambda, \mu)\,(A(\lambda) \otimes A(\mu)) = (A(\mu) \otimes A(\lambda))\, r(\lambda, \mu), \tag{4.1}$$

where

$$r(\lambda, \mu) = \frac{\rho_3\rho_4 - \rho_2^2 + \rho_9\rho_{10}\mathcal{P}}{\rho_4(\rho_3 - \rho_1)}, \tag{4.2}$$

and $\mathcal{P}$ is a 4×4 permutation matrix ($\mathcal{P}x\otimes y = y\otimes x$). If we introduce the reparametrization

$$\begin{aligned} v(\lambda) &= -2\mathrm{i}\,\cot 2\lambda \cosh 2h(\lambda) \\ &= -2\sin k(\lambda) + \frac{\mathrm{i}U}{2} = -2\sin p(\lambda) - \frac{\mathrm{i}U}{2}, \end{aligned} \tag{4.3}$$

the R-matrix $r(\lambda,\mu)$ turns into the rational R-matrix of the isotropic Heisenberg spin chain,

$$r(\lambda,\mu) = \frac{\mathrm{i}U + (v(\lambda)-v(\mu))\mathcal{P}}{\mathrm{i}U + v(\lambda) - v(\mu)}. \tag{4.4}$$

Let us assume that $A(\lambda)$ allows for the following asymptotic expansion in terms of $v(\lambda)$,

$$A(\lambda) = I_2 + \mathrm{i}U \sum_{n=0}^{\infty} \frac{1}{v(\lambda)^{n+1}} \left(\sum_{a=1}^{3} Q_n^a \tilde{\sigma}^a + Q_n^0 I_2 \right), \tag{4.5}$$

where $\tilde{\sigma}^a = G^{ab}\sigma^b$ (cf. (2.23)). Then it follows from general considerations [37, 25, 38] that the first six operators Q_0^a, Q_1^a generate a representation of the Y(su(2)) Yangian quantum group.

There is the following alternative description of the Yangian Y(su(2)) [39]. The Yangian Y(su(2)) is a Hopf algebra which is spanned by six generators Q_n^a ($n = 0, 1,\ a = x, y, z$), satisfying the following relations,

$$\left[Q_0^a, Q_0^b\right] = f^{abc} Q_0^c, \tag{4.6}$$

$$\left[Q_0^a, Q_1^b\right] = f^{abc} Q_1^c, \tag{4.7}$$

$$\begin{aligned} &\left[\left[Q_1^a, Q_1^b\right], \left[Q_0^c, Q_1^d\right]\right] + \left[\left[Q_1^c, Q_1^d\right], \left[Q_0^a, Q_1^b\right]\right] \\ &\quad = \kappa^2 (A^{abkefg} f^{cdk} + A^{cdkefg} f^{abk}) \{Q_0^e, Q_0^f, Q_1^g\}. \end{aligned} \tag{4.8}$$

Here κ is a nonzero constant, $f^{abc} = i\varepsilon^{abc}$ is the antisymmetric tensor of structure constants of su(2), and $A^{abcdef} = f^{adk} f^{bel} f^{cfm} f^{klm}$. The bracket $\{\ \}$ in (4.8) denotes the symmetrized product

$$\{x_1, x_2, x_3\} = \frac{1}{3!} \sum_{\sigma\in S_3} x_{\sigma 1} x_{\sigma 2} x_{\sigma 3}. \tag{4.9}$$

Being a Hopf algebra Y(su(2)) carries an outer structure (co-multiplication, antipode, co-unit), which is described in ref. [39] and which assures that Y(su(2)) has a rich representation theory [40, 41].

A careful consideration of the limit $v(\lambda) \to \infty$ shows that $A(\lambda)$ is indeed of asymptotic form (4.5). There are several possibilities to carry out this limit as a function of λ. We found, however, that only one of these yields finite results for Q_0^a and Q_1^a. We have to take

$\Im(\lambda) \to \infty$ and have to choose the proper branch of solution of eq. (2.5), which determines h as a function of λ. (2.5) implies that

$$\mathrm{e}^{-2h(\lambda)} = -\frac{U}{4}\sin 2\lambda \pm \sqrt{1 + \left(\frac{U}{4}\sin 2\lambda\right)^2}. \tag{4.10}$$

in order to achieve convergence of the matrix elements $\tilde{\mathcal{T}}_{\alpha\beta}$ $(\alpha, \beta = 2, 3)$ we have to choose the lower sign here, then $\mathrm{e}^{-2h(\lambda)}$ is approximately equal to $-\frac{U}{2}\sin 2\lambda$ for large positive values of $\frac{U}{4}\sin 2\lambda$, and we obtain

$$\begin{aligned} \mathrm{e}^{2h(\lambda)} &= \frac{4\mathrm{i}}{U}\mathrm{e}^{2\mathrm{i}\lambda} + O(\mathrm{e}^{6\mathrm{i}\lambda}), \quad \mathrm{e}^{\mathrm{i}k(\lambda)} = \frac{-4}{U}\{\mathrm{e}^{2\mathrm{i}\lambda} + 2\mathrm{e}^{4\mathrm{i}\lambda} + O(\mathrm{e}^{6\mathrm{i}\lambda})\}, \\ \mathrm{e}^{-\mathrm{i}p(\lambda)} &= \frac{4}{U}\{\mathrm{e}^{2\mathrm{i}\lambda} - 2\mathrm{e}^{4\mathrm{i}\lambda} + O(\mathrm{e}^{6\mathrm{i}\lambda})\}, \quad \frac{1}{v(\lambda)} = \frac{-4\mathrm{i}}{U}\{\mathrm{e}^{2\mathrm{i}\lambda} + O(\mathrm{e}^{6\mathrm{i}\lambda})\}. \end{aligned} \tag{4.11}$$

The leading terms in the series (3.11) are of order $\mathrm{e}^{2\mathrm{i}\lambda}$, $\mathrm{e}^{4\mathrm{i}\lambda}, \cdots$. Thus, from the first two sums in (3.11), we get the expansion of the matrix $A(\lambda)$ up to order $\mathrm{e}^{4\mathrm{i}\lambda}$, and the last equation in (4.11) yields the required expansion in $(v(\lambda))^{-1}$ up to second order.

The final result for the representation of Yangian generators is

$$Q_0^a = \frac{1}{2}\sum_j \sigma^a_{\alpha\beta} c^\dagger_{j,\alpha} c_{j,\beta}, \tag{4.12}$$

$$Q_1^a = -\frac{\mathrm{i}}{2}\sum_j \sigma^a_{\alpha\beta} c^\dagger_{j,\alpha}(c_{j+1,\beta} - c_{j-1,\beta}) - \frac{\mathrm{i}U}{4}\sum_{i,j} \mathrm{sgn}(j-i)\sigma^a_{\alpha\beta} c^\dagger_{i,\alpha} c^\dagger_{j,\gamma} c_{i,\gamma} c_{j,\beta}. \tag{4.13}$$

The factor $\mathrm{i}U$ occurring in (4.13) can be identified with the constant κ in (4.8). Note that $Q_0^a = S^a$ is just the operator of the a-component of the total spin (cf. (2.10)). The Yangian representation (4.12) and (4.13) was first obtained by Uglov and Korepin [42, 23]. It can be embedded into a larger family of Yangian representations connected with long-range-hopping extensions of the Hamiltonian (2.1) [43]. Uglov and Korepin showed that Q_0^a and Q_1^a commute with the Hamiltonian on the infinite line.

Since the quantum determinant

$$\mathrm{Det}_q A(\lambda) = A_{11}(\lambda) A_{22}(\check{\lambda}) - A_{12}(\lambda) A_{21}(\check{\lambda}), \tag{4.14}$$

where $v(\check{\lambda}) = v(\lambda) - \mathrm{i}U$, is in the center of the Yangian

$$[\mathrm{Det}_q A(\lambda), A(\mu)] = 0, \tag{4.15}$$

and thus provides a generating function of mutually commuting operators,

$$[\mathrm{Det}_q A(\lambda), \mathrm{Det}_q A(\mu)] = 0, \tag{4.16}$$

it is a natural candidate to generate the Hamiltonian, too. Performing again the asymptotic expansion in terms of $v(\lambda)^{-1}$,

$$\mathrm{Det}_q A(\lambda) = 1 + \mathrm{i}U \sum_{n=0}^{\infty} \frac{J_n}{v(\lambda)^{n+1}}, \tag{4.17}$$

we obtain $J_0 = 0$, $J_1 = \mathrm{i}\hat{H}$, i.e. the Hamiltonian is indeed among the commuting operators generated by $\mathrm{Det}_q A(\lambda)$. All the conserved operators are Yangian invariant by construction. We discuss their relation to the formerly known conserved quantities [16, 19, 44, 45] in section 5.2 below.

In closing this section we shall add a comment. The Hubbard Hamiltonian on the infinite interval is invariant (up to a constant) under the transformation (2.11). The Yangian generators Q_0^a and Q_1^a, however, are transformed into a pair of generators $Q_0^{a\prime}$ and $Q_1^{a\prime}$ of a second, independent representation of Y(su(2)) [42]. These two representations mutually commute. Therefore they can be combined to a direct sum Y(su(2))⊕Y(su(2)). The reason why we get only one of these representations from our QISM approach is that, in order to perform the passage to the infinite interval, we refer to the zero density vacuum $|0\rangle$. This vacuum has lower symmetry than the Hamiltonian. It is invariant under the su(2) Lie algebra of rotations, but does not respect the η-pairing su(2) symmetry of the Hamiltonian. A fully su(2)⊕su(2) invariant vacuum would be the singlet ground state at half filling [46]. It seems to be yet a formidable task to formulate the QISM with respect to this state.

5 Conserved Quantities and Eigenvectors

5.1 Eigenvectors of the Hamiltonian

In the present section we will try to understand the meaning of the operators contained in $B(\lambda), C(\lambda), D(\lambda)$. To begin with, let us have a look at the commutator of $\tilde{\mathcal{T}}(\lambda)$ with the particle number operator $\hat{N}$, which is the extension of $\hat{N}_{nm}$ (2.9) to the infinite interval. Note that $[V(\lambda), \tilde{\Sigma}_\eta^z] = 0$, and thus by (2.25)

$$[\hat{N}, \tilde{\mathcal{T}}(\lambda)] = 2[\tilde{\mathcal{T}}(\lambda), \tilde{\Sigma}_\eta^z]. \tag{5.1}$$

This is a set of 16 equations. Writing out the equations explicitly, we find that $D_{11}(\lambda)$, $D_{22}(\lambda)$ and $A(\lambda)$ conserve the number of particles. The operators $B_{a1}(\lambda)$ and $C_{2a}(\lambda)$ increase

the number of particles by one, whereas $B_{a2}(\lambda)$ and $C_{1a}(\lambda)$ reduce the number of particles by one. $D_{21}(\lambda)$ adds two particles to the system, whereas $D_{12}(\lambda)$ removes two particles. Hence, $B_{a1}(\lambda)$,$C_{2a}(\lambda)$ and $D_{21}(\lambda)$ are candidates for creation operators of eigenstates of the Hamiltonian. The action of these operators on the vacuum follows from (3.11),

$$B_{11}(\lambda)|0\rangle = -\frac{\mathrm{i}\mathrm{e}^{h(\lambda)}}{\cos\lambda}\sum_m \mathrm{e}^{-imp(\lambda)} c_{m\downarrow}^\dagger |0\rangle, \tag{5.2}$$

$$B_{21}(\lambda)|0\rangle = \frac{\mathrm{e}^{h(\lambda)}}{\cos\lambda}\sum_m \mathrm{e}^{-imp(\lambda)} c_{m\uparrow}^\dagger |0\rangle, \tag{5.3}$$

$$C_{21}(\lambda)|0\rangle = -\frac{\mathrm{e}^{-h(\lambda)}}{\cos\lambda}\sum_m \mathrm{e}^{-\mathrm{i}mk(\lambda)} c_{m\uparrow}^\dagger |0\rangle, \tag{5.4}$$

$$C_{22}(\lambda)|0\rangle = -\frac{\mathrm{i}\mathrm{e}^{-h(\lambda)}}{\cos\lambda}\sum_m \mathrm{e}^{-\mathrm{i}mk(\lambda)} c_{m\downarrow}^\dagger |0\rangle, \tag{5.5}$$

$$D_{21}(\lambda)|0\rangle = \frac{\mathrm{i}}{\cos^2\lambda}\sum_{m,n} c_{m\uparrow}^\dagger c_{n\downarrow}^\dagger \left\{\theta(m \geq n)\mathrm{e}^{-\mathrm{i}(mk(\lambda)+np(\lambda))} + \theta(m < n)\mathrm{e}^{-\mathrm{i}(nk(\lambda)+mp(\lambda))}\right\}|0\rangle. \tag{5.6}$$

These states are the most elementary eigenstates of the Hamiltonian as can be verified by direct calculation. Note that they are bounded in different, disconnected parts of the spectral parameter space. This is because of the constraint (2.5), which turns into

$$\sin k(\lambda) - \sin p(\lambda) = \frac{\mathrm{i}U}{2} \tag{5.7}$$

when expressed in terms of $k(\lambda)$, $p(\lambda)$ (3.14). If $k(\lambda)$ is real, $p(\lambda)$ cannot be real for non-zero U and vice versa. $D_{21}(\lambda)$ creates a bound state, if $\Im(k) = -\Im(p) < 0$. This condition is compatible with the constraint (5.7) on $k(\lambda)$ and $p(\lambda)$. Let $U \neq 0$. Then $\kappa := \Im(p) = -\Im(k)$ implies $q := \Re(k) = \Re(p) \bmod 2\pi$, and

$$\sinh\kappa = -\frac{U}{4\cos q}. \tag{5.8}$$

Hence, if $U < 0$, $D_{21}(\lambda)$ creates a bound state for $|q| < \pi/2$, and if $U > 0$, $D_{21}(\lambda)$ creates a bound state for $\pi/2 < |q| < \pi$.

The commutators of the various operators contained in $\tilde{\mathcal{T}}(\lambda)$ with $\mathrm{Det}_q(A(\mu))$ are summarized in Appendix B. Let us insert the asymptotic expansion (4.17) into (B.23)-(B.28). Then there occur only two different ratios of Boltzmann weights in these equations, which may be expanded by use of (4.11) as

$$-\frac{\mathrm{i}\rho_1(\lambda,\mu)}{\rho_9(\lambda,\mu)} = 1 + \frac{\mathrm{i}U}{2v(\mu)} + \left(\frac{\mathrm{i}U}{8} - \mathrm{i}\mathrm{e}^{-\mathrm{i}k(\lambda)}\right)\frac{\mathrm{i}U}{v(\mu)^2} + O\left(\frac{1}{v(\mu)^3}\right), \tag{5.9}$$

$$\frac{\mathrm{i}\rho_{10}(\lambda,\mu)}{\rho_4(\lambda,\mu)} = 1 + \frac{\mathrm{i}U}{2v(\mu)} + \left(\frac{\mathrm{i}U}{8} + \mathrm{i}\mathrm{e}^{\mathrm{i}p(\lambda)}\right)\frac{\mathrm{i}U}{v(\mu)^2} + O\left(\frac{1}{v(\mu)^3}\right). \tag{5.10}$$

Comparing terms of second order in $v(\mu)^{-1}$, we obtain the following commutators,

$$
\begin{aligned}
[\hat{H}, B_{a1}\lambda)] &= -(2\cos p(\lambda) + U/2)B_{a1}(\lambda), && (5.11)\\
[\hat{H}, B_{a2}(\lambda)] &= (2\cos k(\lambda) + U/2)B_{a2}(\lambda), && (5.12)\\
[\hat{H}, C_{1a}\lambda)] &= (2\cos p(\lambda) + U/2)C_{1a}(\lambda), && (5.13)\\
[\hat{H}, C_{2a}(\lambda)] &= -(2\cos k(\lambda) + U/2)C_{2a}(\lambda), && (5.14)\\
[\hat{H}, D_{12}(\lambda)] &= 2(\mathrm{e}^{\mathrm{i}p(\lambda)} + \mathrm{e}^{-\mathrm{i}k(\lambda)})D_{12}(\lambda), && (5.15)\\
[\hat{H}, D_{21}(\lambda)] &= -2(\mathrm{e}^{\mathrm{i}p(\lambda)} + \mathrm{e}^{-\mathrm{i}k(\lambda)})D_{21}(\lambda). && (5.16)
\end{aligned}
$$

The above results justify the interpretation of $B_{a1}(\lambda)$,$C_{2a}(\lambda)$ and $D_{21}(\lambda)$ as creation operators. $B_{a1}(\lambda)$ and $C_{2a}(\lambda)$ create single particle excitations, whereas $D_{21}(\lambda)$ creates a bound state of two particles. Let us investigate several examples. Since $\hat{H}|0\rangle = 0$, (5.14) implies, for instance,

$$\hat{H}C_{2a}(\lambda)|0\rangle = -(2\cos k(\lambda) + U/2)\, C_{2a}(\lambda)|0\rangle, \tag{5.17}$$

or more generally

$$\hat{H}C_{2a_1}(\lambda_1)\dots C_{2a_n}(\lambda_n)|0\rangle = -\sum_{j=1}^{n}(2\cos k(\lambda_j) + U/2)\, C_{2a_1}(\lambda_1)\dots C_{2a_n}(\lambda_n)|0\rangle. \tag{5.18}$$

A similar result holds for states, where the operators $B_{1a}(\lambda)$ or mixed products of operators $B_{a1}(\lambda)$ and $C_{2a}(\lambda)$ are applied to the vacuum. We have to remember here that $B_{a1}(\lambda)$ and $C_{2a}(\mu)$ create bounded states, only if $p(\lambda)$ and $k(\mu)$ are real. The restrictions on $k(\lambda)$ and $p(\lambda)$ occurring in (5.16) have been, $k = q - \mathrm{i}\kappa$, $p = q + \mathrm{i}\kappa$, where q is real and $\kappa > 0$. Taking these restrictions into account we obtain, for instance,

$$\hat{H}D_{21}(\lambda)|0\rangle = -(4\cos q\cosh\kappa + U)\, D_{21}(\lambda)|0\rangle. \tag{5.19}$$

The constraint (5.7) implies

$$4\cos q\cosh\kappa = \pm\sqrt{16\cos^2 q + U^2}, \tag{5.20}$$

where the plus sign has to be taken for $U < 0$, $|q| < \frac{\pi}{2}$, and the minus sign is relevant, if $U > 0$ and $\frac{\pi}{2} < |q| < \pi$. This is, of course, in accordance with intuition.

5.2 Higher Conserved Quantities

Within the present formalism it is of course more natural to consider the quantum determinant $\mathrm{Det}_q(A(\lambda))$ rather than the Hamiltonian. It can be seen from the commutation relations

in Appendix B that arbitrary products of operators $B_{a1}(\lambda)$, $C_{2a}(\lambda)$ and $D_{21}(\lambda)$ create eigenstates of the quantum determinant $\mathrm{Det}_q A(\lambda)$. For example, (B.26) and $\mathrm{Det}_q A(\mu)|0\rangle = |0\rangle$ yield

$$\mathrm{Det}_q(A(\mu))C_{2a}(\lambda)|0\rangle = -\frac{\rho_1(\lambda,\mu)\rho_1(\lambda,\check{\mu})}{\rho_9(\lambda,\mu)\rho_9(\lambda,\check{\mu})}\frac{v(\lambda)-v(\mu)+\mathrm{i}U}{v(\lambda)-v(\mu)}C_{2a}(\lambda)|0\rangle, \tag{5.21}$$

where $\check{\mu}$ is defined by $v(\check{\mu}) = v(\mu) - \mathrm{i}U$. The eigenvalue of multi-particle states is a product of eigenvalues of single particle states. For this reason $\ln \mathrm{Det}_q(A(\mu))$ is a generating function of conserved operators, which is possessing an additive spectrum on multi-particle states.

Commuting conserved operators for the Hubbard model have been constructed by several authors [16, 19, 44, 45] either by use of ad hoc methods or by using the expansion (2.8). We present the first few known of them in Appendix D. Here we ask for the relation of these known conserved operators to the ones generated by $\ln \mathrm{Det}_q(A(\mu))$. To this end let us compare the action on one-particle states. Equation (5.21) implies

$$\ln \mathrm{Det}_q(A(\mu))\sum_j \mathrm{e}^{-\mathrm{i}kj}c_{j\sigma}^\dagger|0\rangle = \Upsilon(\mu,k)\sum_j \mathrm{e}^{-\mathrm{i}kj}c_{j\sigma}^\dagger|0\rangle, \tag{5.22}$$

where

$$\Upsilon(\mu,k) = \ln\left[\frac{\cos\dfrac{k+p(\mu)}{2}}{\sin\dfrac{k-k(\mu)}{2}}\;\frac{\cos\dfrac{k+p(\check{\mu})}{2}}{\sin\dfrac{k-k(\check{\mu})}{2}}\;\frac{\sin k-\sin k(\mu)-\mathrm{i}U/2}{\sin k-\sin k(\mu)}\right]. \tag{5.23}$$

Comparing the expansion of $\Upsilon(\mu,k)$ in terms of $v(\lambda)^{-1}$ with the eigenvalues of the first few explicitly known higher conserved quantities $H_1(=\hat{H})$, H_2, H_3, H_4 (for details see Appendix D), we are led to the following conjecture,

$$\begin{aligned}
&\ln \mathrm{Det}_q(A(\mu))\\
&\quad= \frac{\mathrm{i}U}{v(\mu)^2}\mathrm{i}H_1 + \frac{\mathrm{i}U}{v(\mu)^3}(\mathrm{i}H_2 - UH_1)\\
&\quad+ \frac{\mathrm{i}U}{v(\mu)^4}\left[\mathrm{i}H_3 - \frac{3U}{2}H_2 + \left(\frac{-5\mathrm{i}}{4}U^2 + 3\mathrm{i}\right)H_1\right]\\
&\quad+ \frac{\mathrm{i}U}{v(\mu)^5}\left[\mathrm{i}H_4 - 2UH_3 + \left(-\frac{11\mathrm{i}}{4}U^2 + 4\mathrm{i}\right)H_2 + \left(\frac{3U^3}{2} - 6U\right)H_1\right]\\
&\quad+ O\left(\frac{1}{v(\mu)^6}\right).
\end{aligned} \tag{5.24}$$

The coefficients of the asymptotic expansion of $\ln \mathrm{Det}_q(A(\mu))$ are linear combinations of the formerly known conserved quantities. Particularly, they do not give a complete set of conserved operators of the free fermion model in the limit $U \to 0$ (cf. Appendix D). It

remains therefore an open question, if $\ln \mathrm{Det}_q(A(\lambda))$ generates a complete set of commuting operators for the Hubbard model or not. Another fact which can be concluded from (5.24) is the Yangian invariance of the operators H_s (cf. [47]).

At this point a remark on the construction of Dunkl operators may be in order. Dunkl operators are commuting difference (differential) operators, which turned out to be useful in the investigation of exactly solvable long-range interacting systems [48, 38]. More precisely, a set of difference operators $\{d_j\}$ is a set of Dunkl operators, if there exists a representation $\{K_{ij}\}$ of the symmetric group, such that the operators d_j, K_{ij} form a representation of the degenerate affine Hecke algebra. Given a representation of the degenerate affine Hecke algebra, it is possible to construct a corresponding representation of the Y(su(2)) Yangian out of it [48, 38]. The strategy developed in ref. [48, 38] was successfully applied to the fermionic nonlinear Schrödinger model [25]. Yet it seems to be inappropriate for the Hubbard model for the following reason. Let P_{0j} be a permutation operator acting on su(p) spins. Then it follows from the defining relations of the degenerate affine Hecke algebra that the transfer matrix

$$\hat{T}_0(u) = \left(1 + \frac{\mathrm{i}c\ P_{01}}{u - d_1}\right) \cdots \left(1 + \frac{\mathrm{i}c\ P_{0n}}{u - d_n}\right) \tag{5.25}$$

preserves the space of fermionic wave functions. It satisfies the Yang-Baxter relations with R-matrix $u + \mathrm{i}cP$ and thus generates a representation of Y(su(p)). In case of the fermionic nonlinear Schrödinger model the quantum determinant $\mathrm{Det}_q(\hat{T}(u))$ agrees with the quantum determinant of a submatrix of the monodromy matrix obtained within the quantum inverse scattering approach [25]. It has the eigenvalue

$$\prod_{j=1}^{n} \left(1 + \frac{\mathrm{i}c}{u - k_j}\right), \tag{5.26}$$

By way of contrast the eigenvalue of the quantum determinant $\mathrm{Det}_q(A(\mu))$ is

$$\prod_{j=1}^{n} \exp \Upsilon(\mu, k_j) = \prod_{j=1}^{n} \left[\frac{\cos \dfrac{k_j + p(\mu)}{2}}{\sin \dfrac{k_j - k(\mu)}{2}} \ \frac{\cos \dfrac{k_j + p(\check{\mu})}{2}}{\sin \dfrac{k_j - k(\check{\mu})}{2}} \ \frac{\sin k_j - \sin k(\mu) - \mathrm{i}U/2}{\sin k_j - \sin k(\mu)} \right]. \tag{5.27}$$

Hence it seems that the method developed in ref. [25] has to be modified, if we want to apply it to the Hubbard model.

5.3 The elements of the monodromy matrix under Yangian transformations

As we have shown above, the submatrix $A(\lambda)$ of the monodromy matrix $\tilde{\mathcal{T}}(\lambda)$ generates a representation of Y(su(2)). Let us look for the commutators of the remaining elements of the monodromy matrix, which can be arranged in submatrices $B(\lambda)$, $C(\lambda)$, $D(\lambda)$, with the Yangian generators Q_n^a, $n = 0, 1$; $a = x, y, z$. Combining (4.5) and (B.8)-(B.14) we end up with

$$\begin{aligned}
[Q_0^a, B(\lambda)] &= -\frac{1}{2}\tilde{\sigma}^a B(\lambda), && (5.28)\\
[Q_1^a, B(\lambda)] &= \sin p(\lambda)\tilde{\sigma}^a B(\lambda) + \frac{U}{2}\varepsilon^{abc}\tilde{\sigma}^b B(\lambda) Q_0^c, && (5.29)\\
[Q_0^a, C(\lambda)] &= \frac{1}{2}C(\lambda)\tilde{\sigma}^a, && (5.30)\\
[Q_1^a, C(\lambda)] &= -\sin k(\lambda) C(\lambda)\tilde{\sigma}^a + \frac{U}{2}\varepsilon^{abc} C(\lambda)\tilde{\sigma}^b Q_0^c, && (5.31)\\
[Q_0^a, D(\lambda)] &= [Q_1^a, D(\lambda)] \;=\; 0. && (5.32)
\end{aligned}$$

In the next section we will see that these equations determine the behavior of the eigenstates of the Hamiltonian under Yangian transformations. The discussion of the irreducible representations on the subspace of a fixed number of one-particle excitations created by $C_{2a}(\lambda)$ can be done in analogy with ref. [25].

6 Construction of n-particle states

6.1 Scattering States

We have seen in the preceding section that the repeated action of operators $B_{1a}(\lambda)$, $C_{2a}(\lambda)$ on the vacuum produces n-particle eigenstates of the quantum determinant of $A(\lambda)$. For small enough n the corresponding wave functions can be worked out by hand. They are of the form of Bethe wave functions and are easily understood as scattering states of n-particles. For scattering states there is a natural normalization. We have to require the amplitude of the incident wave to be unity. In previously studied cases [26, 27, 49] it turned out that such kind of normalization was obtainable by introducing the operator analog of the reflection coefficient of the corresponding classical problem. In other words, the creation operators were normalized by multiplying with the inverse of certain generators of conserved quantities.

For the Hubbard model we propose the following two pairs of normalized creation operators,

$$
\begin{aligned}
R_\alpha(\lambda)^\dagger &= \mathrm{i}^{3-\alpha}\mathrm{e}^{h(\lambda)}\cos\lambda\ C_{2\alpha}(\lambda)D_{22}(\lambda)^{-1}\ (\alpha=1,2), && (6.1)\\
\hat{R}_\alpha(\lambda)^\dagger &= \mathrm{i}^{\alpha-1}\mathrm{e}^{-h(\lambda)}\cos\lambda\ B_{3-\alpha,1}(\lambda)D_{11}(\lambda)^{-1}\ (\alpha=1,2). && (6.2)
\end{aligned}
$$

In these formulae $\alpha=1$ corresponds to spin-up and $\alpha=2$ to spin-down, respectively. The numerical prefactors have been obtained by demanding that $R_\alpha(\lambda)^\dagger$ and $\hat{R}_\alpha(\lambda)^\dagger$ generate normalized one-particle states above the vacuum,

$$
R_\alpha(\lambda)^\dagger|0\rangle = \sum_m \mathrm{e}^{-imk(\lambda)}c^\dagger_{m,\alpha}|0\rangle, \quad \hat{R}_\alpha(\lambda)^\dagger|0\rangle = \sum_m \mathrm{e}^{-imp(\lambda)}c^\dagger_{m,\alpha}|0\rangle. \qquad (6.3)
$$

Hereafter we assume that λ is chosen in such a way that $R_\alpha(\lambda)^\dagger$ and $\hat{R}_\alpha(\lambda)^\dagger$ create physical states. This means for $R_\alpha(\lambda)^\dagger$ that $k(\lambda)$ has to be real and for $\hat{R}_\alpha(\lambda)^\dagger$ that $p(\lambda)$ has to be real.

It is not difficult to see that eqs. (2.29)-(2.32), which determine the behavior of the elements of the monodromy matrix under hermitian conjugation, remain valid in the thermodynamic limit. Hence, they can be used to obtain the conjugated annihilation operators corresponding to $R_\alpha(\lambda)^\dagger$ and $\hat{R}_\alpha(\lambda)^\dagger$,

$$
\begin{aligned}
R_\alpha(\lambda) &= \mathrm{i}^{2-\alpha}\mathrm{e}^{h(\lambda')}\sin\lambda'\ D_{11}(\lambda')^{-1}C_{1,3-\alpha}(\lambda')\ (\alpha=1,2), && (6.4)\\
\hat{R}_\alpha(\lambda) &= \mathrm{i}^{\alpha-2}\mathrm{e}^{-h(\lambda')}\sin\lambda'\ D_{22}(\lambda')^{-1}B_{\alpha 2}(\lambda')\ (\alpha=1,2), && (6.5)
\end{aligned}
$$

where $\lambda'=\pi/2-\lambda^*$. The commutation relations between the normalized operators are easily calculated by use of the formulae presented in Appendix B. Provided $\lambda\neq\mu$ (mod 2π) the results are

$$
\begin{aligned}
R_\alpha(\lambda)^\dagger R_\beta(\mu)^\dagger &= -r(\lambda,\mu)_{\gamma\delta,\alpha\beta}R_\gamma(\mu)^\dagger R_\delta(\lambda)^\dagger, && (6.6)\\
R_\alpha(\lambda) R_\beta(\mu)^\dagger &= -r(\mu,\lambda)_{\gamma\alpha,\delta\beta}R_\gamma(\mu)^\dagger R_\delta(\lambda), && (6.7)\\
\hat{R}_\alpha(\lambda)^\dagger \hat{R}_\beta(\mu)^\dagger &= -r(\mu,\lambda)_{\gamma\delta,\alpha\beta}\hat{R}_\gamma(\mu)^\dagger \hat{R}_\delta(\lambda)^\dagger, && (6.8)\\
\hat{R}_\alpha(\lambda) \hat{R}_\beta(\mu)^\dagger &= -r(\lambda,\mu)_{\gamma\alpha,\delta\beta}\hat{R}_\gamma(\mu)^\dagger \hat{R}_\delta(\lambda), && (6.9)\\
R_\alpha(\lambda)^\dagger \hat{R}_\beta(\mu)^\dagger &= -\hat{R}_\beta(\mu)^\dagger R_\alpha(\lambda)^\dagger, && (6.10)\\
R_\alpha(\lambda) \hat{R}_\beta(\mu)^\dagger &= -\hat{R}_\beta(\mu)^\dagger R_\alpha(\lambda). && (6.11)
\end{aligned}
$$

The operators $R_\alpha(\lambda)$, $R_\alpha(\lambda)^\dagger$ and $\hat{R}_\alpha(\lambda)$, $\hat{R}_\alpha(\lambda)^\dagger$ form a representation of the graded Zamolodchikov-Faddeev algebra with S-matrix $r(\lambda,\mu)$. These representations may be identified as representations of left and right Zamolodchikov-Faddeev algebra, respectively [50, 26,

51, 6, 52]. The grading is such that all operators are odd. In physical terms we may say that $R_\alpha(\lambda)^\dagger$ and $\hat{R}_\alpha(\lambda)^\dagger$ are creation operators of fermionic quasi-particles. Both quasi-particles are charge density waves, because $R_\alpha(\lambda)^\dagger$ and $\hat{R}_\alpha(\lambda)^\dagger$ add a particle to the system. Note that we cannot have spin density waves over the zero density vacuum.

It is possible to calculate the S-matrix for two-body scattering processes within the coordinate Bethe ansatz from two-body phase shifts [53, 54]. This approach was applied to the Hubbard model at half-filling by Eßler and Korepin [46]. The presence of a finite density background of particles influences the scattering. Thus the S-matrix of Eßler and Korepin for holon-holon scattering differs from ours by a dressing factor.

Let us make our above statements about the creation of normalized scattering states more precise. We shall present the two-particle states generated by $R_\alpha(\lambda)^\dagger$ or $\hat{R}_\alpha(\lambda)^\dagger$ as derived from (3.11) and the commutation relations between the elements of the monodromy matrix.

$$R_1(\lambda)^\dagger R_1(\mu)^\dagger|0\rangle = \sum_{n,m} c_{n\uparrow}^\dagger c_{m\uparrow}^\dagger \mathrm{e}^{-\mathrm{i}nk(\lambda)}\mathrm{e}^{-\mathrm{i}mk(\mu)}|0\rangle, \tag{6.12}$$

$$R_2(\lambda)^\dagger R_2(\mu)^\dagger|0\rangle = \sum_{n,m} c_{n\downarrow}^\dagger c_{m\downarrow}^\dagger \mathrm{e}^{-\mathrm{i}nk(\lambda)}\mathrm{e}^{-\mathrm{i}mk(\mu)}|0\rangle, \tag{6.13}$$

$$\begin{aligned} R_1(\lambda)^\dagger R_2(\mu)^\dagger|0\rangle &= \sum_{n,m} c_{n\uparrow}^\dagger c_{m\downarrow}^\dagger \Bigg[\theta(n \geq m)\mathrm{e}^{-\mathrm{i}nk(\lambda)}\mathrm{e}^{-\mathrm{i}mk(\mu)}\frac{v(\lambda)-v(\mu)}{v(\lambda)-v(\mu)+\mathrm{i}U} \\ &+\theta(n<m)\mathrm{e}^{-\mathrm{i}nk(\lambda)}\mathrm{e}^{-\mathrm{i}mk(\mu)} + \theta(n<m)\mathrm{e}^{-\mathrm{i}mk(\lambda)}\mathrm{e}^{-\mathrm{i}nk(\mu)}\frac{-\mathrm{i}U}{v(\lambda)-v(\mu)+\mathrm{i}U}\Bigg]|0\rangle, \end{aligned} \tag{6.14}$$

$$\begin{aligned} R_2(\lambda)^\dagger R_1(\mu)^\dagger|0\rangle &= \sum_{n,m} c_{n\downarrow}^\dagger c_{m\uparrow}^\dagger \Bigg[\theta(n \geq m)\mathrm{e}^{-\mathrm{i}nk(\lambda)}\mathrm{e}^{-\mathrm{i}mk(\mu)}\frac{v(\lambda)-v(\mu)}{v(\lambda)-v(\mu)+\mathrm{i}U} \\ &+\theta(n<m)\mathrm{e}^{-\mathrm{i}nk(\lambda)}\mathrm{e}^{-\mathrm{i}mk(\mu)} + \theta(n<m)\mathrm{e}^{-\mathrm{i}mk(\lambda)}\mathrm{e}^{-\mathrm{i}nk(\mu)}\frac{-\mathrm{i}U}{v(\lambda)-v(\mu)+\mathrm{i}U}\Bigg]|0\rangle, \end{aligned} \tag{6.15}$$

$$\hat{R}_1(\lambda)^\dagger \hat{R}_1(\mu)^\dagger|0\rangle = \sum_{n,m} c_{n\uparrow}^\dagger c_{m\uparrow}^\dagger \mathrm{e}^{-\mathrm{i}np(\lambda)}\mathrm{e}^{-\mathrm{i}mp(\mu)}|0\rangle, \tag{6.16}$$

$$\hat{R}_2(\lambda)^\dagger \hat{R}_2(\mu)^\dagger|0\rangle = \sum_{n,m} c_{n\downarrow}^\dagger c_{m\downarrow}^\dagger \mathrm{e}^{-\mathrm{i}np(\lambda)}\mathrm{e}^{-\mathrm{i}mp(\mu)}|0\rangle, \tag{6.17}$$

$$\begin{aligned} \hat{R}_1(\lambda)^\dagger \hat{R}_2(\mu)^\dagger|0\rangle &= \sum_{n,m} c_{n\uparrow}^\dagger c_{m\downarrow}^\dagger \Bigg[\theta(n \leq m)\mathrm{e}^{-\mathrm{i}np(\lambda)}\mathrm{e}^{-\mathrm{i}mp(\mu)}\frac{v(\lambda)-v(\mu)}{v(\lambda)-v(\mu)-\mathrm{i}U} \\ &+\theta(n>m)\mathrm{e}^{-\mathrm{i}np(\lambda)}\mathrm{e}^{-\mathrm{i}mp(\mu)} + \theta(n>m)\mathrm{e}^{-\mathrm{i}mp(\lambda)}\mathrm{e}^{-\mathrm{i}np(\mu)}\frac{\mathrm{i}U}{v(\lambda)-v(\mu)-\mathrm{i}U}\Bigg]|0\rangle, \end{aligned} \tag{6.18}$$

$$\begin{aligned} \hat{R}_2(\lambda)^\dagger \hat{R}_1(\mu)^\dagger|0\rangle &= \sum_{n,m} c_{n\downarrow}^\dagger c_{m\uparrow}^\dagger \Bigg[\theta(n \leq m)\mathrm{e}^{-\mathrm{i}np(\lambda)}\mathrm{e}^{-\mathrm{i}mp(\mu)}\frac{v(\lambda)-v(\mu)}{v(\lambda)-v(\mu)-\mathrm{i}U} \\ &+\theta(n>m)\mathrm{e}^{-\mathrm{i}np(\lambda)}\mathrm{e}^{-\mathrm{i}mp(\mu)} + \theta(n>m)\mathrm{e}^{-\mathrm{i}mp(\lambda)}\mathrm{e}^{-\mathrm{i}np(\mu)}\frac{\mathrm{i}U}{v(\lambda)-v(\mu)-\mathrm{i}U}\Bigg]|0\rangle. \end{aligned} \tag{6.19}$$

Note that the two-particle states (6.12)-(6.15) generated by $R_\alpha(\lambda)^\dagger$ are in-states if $k(\lambda) < k(\mu)$ and out-states if $k(\lambda) > k(\mu)$. Moreover, they are normalized in the sense explained above. As for the operators $\hat{R}_\alpha(\lambda)^\dagger$ we observe similar things. The two-particle states (6.16)-(6.19) are normalized in-states if $p(\lambda) > p(\mu)$ and normalized out-states if $p(\lambda) < p(\mu)$. These facts, together with the examples of other integrable models [27, 49] lead us to the following conjecture:

Conjecture 1 *Provided $k(\lambda_j)$ is real for $j = 1, \cdots, n$, the n-particle state*

$$R_{\alpha_1}(\lambda_1)^\dagger \cdots R_{\alpha_n}(\lambda_n)^\dagger |0\rangle \tag{6.20}$$

is a normalized in-state if $k(\lambda_1) < \cdots < k(\lambda_n)$ and a normalized out-state if $k(\lambda_1) > \cdots > k(\lambda_n)$.
Provided $p(\mu_j)$ is real for $j = 1, \cdots, n$, the n-particle state

$$\hat{R}_{\alpha_1}(\mu_1)^\dagger \cdots \hat{R}_{\alpha_n}(\mu_n)^\dagger |0\rangle \tag{6.21}$$

is a normalized in-state if $p(\mu_1) > \cdots > p(\mu_n)$ and a normalized out-state if $p(\mu_1) < \cdots < p(\mu_n)$.

The proof of this conjecture seems difficult for general n, since it seems to be unavoidable to use the series (3.11) and the explicit form (3.13) of $\tilde{\mathcal{L}}_m(\lambda)$.

We have two pairs of normalized one-particle creation operators now, but as in the case of $B_{a1}(\lambda)$ and $C_{2a}(\lambda)$, we do not need to care about both of them in constructing multi-particle states. We may use the operator $R_\alpha(\lambda)^\dagger$ only (or $\hat{R}_\alpha(\lambda)$ only). The reason is the following. From (6.3) we deduce that

$$\hat{R}_\alpha(\lambda)^\dagger |0\rangle = R_\alpha(\tilde{\lambda})^\dagger |0\rangle, \tag{6.22}$$

where $p(\lambda) = k(\tilde{\lambda})$. Hence the action of a mixed product of $R^\dagger$ and $\hat{R}^\dagger$ on the vacuum can be expressed in the form (6.20) by use of (6.10) and (6.22). In particular, one easily obtains

$$\hat{R}_{\alpha_n}(\lambda_n)^\dagger \cdots \hat{R}_{\alpha_1}(\lambda_1)^\dagger |0\rangle = (-1)^{\frac{n(n-1)}{2}} R_{\alpha_1}(\tilde{\lambda}_1)^\dagger \cdots R_{\alpha_n}(\tilde{\lambda}_n)^\dagger |0\rangle, \tag{6.23}$$

where $p(\lambda_j) = k(\tilde{\lambda}_j)$. The order of the operators is reversed when written in terms of $R^\dagger_\alpha(\lambda)$ instead of $\hat{R}^\dagger_\alpha(\lambda)$.

It turns out that the introduction of the prefactors in eqs. (6.1) and (6.2) also removes the twist (2.22) from the commutators of our redefined creation and annihilation operators

with the Yangian generators. Using (5.28)-(5.32) and some of the formulae in Appendix B, we obtain

$$[Q_0^a, R_\alpha(\lambda)^\dagger] = \frac{1}{2} R_\beta(\lambda)^\dagger \sigma^a_{\beta\alpha}, \tag{6.24}$$

$$[Q_1^a, R_\alpha(\lambda)^\dagger] = -\sin k(\lambda) R_\beta(\lambda)^\dagger \sigma^a_{\beta\alpha} + \frac{U}{2}\varepsilon^{abc} R_\beta(\lambda)^\dagger \sigma^b_{\beta\alpha} Q_0^c, \tag{6.25}$$

$$[Q_0^a, \hat{R}_\alpha(\lambda)^\dagger] = \frac{1}{2} \hat{R}_\beta(\lambda)^\dagger \sigma^a_{\beta\alpha}, \tag{6.26}$$

$$[Q_1^a, \hat{R}_\alpha(\lambda)^\dagger] = -\sin p(\lambda) \hat{R}_\beta(\lambda)^\dagger \sigma^a_{\beta\alpha} - \frac{U}{2}\varepsilon^{abc} \hat{R}_\beta(\lambda)^\dagger \sigma^b_{\beta\alpha} Q_0^c. \tag{6.27}$$

These formulae induce an adjoint action of the Yangian on n-particle states [55, 25]. Noting that $Q_0^a|0\rangle = 0 = Q_1^a|0\rangle$, we obtain the action of the Yangian on the $n = 1$ sector as

$$Q_0^a R_\alpha(\lambda)^\dagger|0\rangle = \frac{1}{2}\sigma^a_{\beta\alpha} R_\beta(\lambda)^\dagger|0\rangle, \tag{6.28}$$

$$Q_1^a R_\alpha(\lambda)^\dagger|0\rangle = -\sin k(\lambda) \sigma^a_{\beta\alpha} R_\beta(\lambda)^\dagger|0\rangle. \tag{6.29}$$

Since the action of Q_1^a is $-2\sin k(\lambda)$ times that of Q_0^a, the representation is called the fundamental representation $W_1(-2\sin k(\lambda))$ [40, 41].

In the two-particle sector ($n = 2$) we get

$$Q_0^a R_\alpha(\lambda_1)^\dagger R_\sigma(\lambda_2)^\dagger|0\rangle = \left(\frac{1}{2}\sigma^a_{\beta\alpha}\delta_{\rho\sigma} + \frac{1}{2}\delta_{\beta\alpha}\sigma^a_{\rho\sigma}\right) R_\beta(\lambda_1)^\dagger R_\rho(\lambda_2)^\dagger|0\rangle, \tag{6.30}$$

$$\begin{aligned} Q_1^a R_\alpha(\lambda_1)^\dagger R_\sigma(\lambda_2)^\dagger|0\rangle &= \left(-\sin k(\lambda_1)\sigma^a_{\beta\alpha}\delta_{\rho\sigma} - \sin k(\lambda_2)\delta_{\beta\alpha}\sigma^a_{\rho\sigma} + \frac{U}{4}\varepsilon^{abc}\sigma^b_{\beta\alpha}\sigma^c_{\rho\sigma}\right) \\ &\quad \cdot R_\beta(\lambda_1)^\dagger R_\rho(\lambda_2)^\dagger|0\rangle. \end{aligned} \tag{6.31}$$

This representation is a tensor product representation $W_1(-2\sin k(\lambda_1)) \otimes W_1(-2\sin k(\lambda_2))$ with co-multiplication Δ defined by

$$\Delta(Q_0^a) = Q_0^a \otimes 1 + 1 \otimes Q_0^a, \tag{6.32}$$

$$\Delta(Q_1^a) = Q_1^a \otimes 1 + 1 \otimes Q_1^a + U\varepsilon^{abc} Q_0^b \otimes Q_0^c. \tag{6.33}$$

It is four-dimensional and irreducible, since $k(\lambda_1)$ and $k(\lambda_2)$ are real. Due to the Yangian invariance of the Hamiltonian, these four states are degenerate. Under the sub-algebra su(2) of spins this multiplet is decoupled to su(2)-triplet and su(2)-singlet. We can say that the Yangian Y(su(2)) mixes spin multiplets to form a larger multiplet.

Similarly, the n-particle states $R_{\alpha_1}(\lambda_1)^\dagger \cdots R_{\alpha_n}(\lambda_n)^\dagger|0\rangle$ ($\alpha_j = 1, 2$) transform under Y(su(2)) as tensor product representations $W_1(-2\sin k(\lambda_1)) \otimes \cdots \otimes W_1(-2\sin k(\lambda_n))$. These

representations are irreducible under the Yangian Y(su(2)), since the quasi-momenta $k(\lambda_j)$ are real, but not irreducible under the sub-algebra su(2). In other words, these 2^n states form a multiplet under Y(su(2)), while under su(2) they decay into some multiplets according to the value of the total spin. Thus su(2) is not sufficient to explain the large degeneracy of the system in the thermodynamic limit.

The irreducibility leads us to the conclusion that we can construct all the n-particle states (6.20) out of the Yangian highest weight state

$$R_1(\lambda_1)^\dagger \cdots R_1(\lambda_n)^\dagger |0\rangle, \tag{6.34}$$

by acting with Yangian generators Q_n^a. The wave function of the above state (6.34) must be of plane-wave form, since the on-site interaction never occurs between up-spin particles due to the Pauli principle. Therefore, assuming that the state (6.34) is a normalized one (see Conjecture 1), we conjecture that the above state (6.34) is equal to the state

$$c_\uparrow^\dagger(k(\lambda_1)) \cdots c_\uparrow^\dagger(k(\lambda_n))|0\rangle, \tag{6.35}$$

where $c_\sigma^\dagger(k) = \sum_j c_{j\sigma}^\dagger \mathrm{e}^{-\mathrm{i}jk}$. Thus we have got a simple method for constructing multi-particle scattering states. They are obtained out of the plane-wave state (6.35) by using the Yangian generators (4.12) and (4.13). We have already encountered such kind of situation in the case of the repulsive δ-function fermi gas [25].

One can similarly discuss Yangian representations of multi-particle states constructed by use of $\hat{R}_\alpha^\dagger(\lambda)$. The alert reader will have noticed the different signs in front of U in eqs. (6.25) and (6.27), which lead to different definitions of the co-multiplication (cf. (6.32), (6.33)):

$$\Delta'(Q_0^a) \quad = \quad Q_0^a \otimes 1 + 1 \otimes Q_0^a, \tag{6.36}$$

$$\Delta'(Q_1^a) \quad = \quad Q_1^a \otimes 1 + 1 \otimes Q_1^a - U\varepsilon^{abc} Q_0^b \otimes Q_0^c. \tag{6.37}$$

But this does not cause any contradiction. The order of the quasi-momenta in (6.23) is reversed in the multi-particle states expressed by $R_\alpha(\lambda)^\dagger$ compared to those expressed by $\hat{R}_\alpha(\lambda)^\dagger$. This corresponds to the reversed order of the tensor product $\otimes$ in the definition of the co-multiplication, which compensates the different sign in front of U in (6.33) and (6.37).

The various parameters U, λ, h, v, k and p which we used so far are connected through the formulae (2.5) and (4.3). Thus only two of them are independent. As a test of consistency of the results in this section let us consider the free fermion limit $U \to 0$. This limit is most

conveniently taken for fixed v, for if we fix $v = v(\lambda)$ and $\bar{v} = v(\mu)$ in eqs. (6.12)-(6.19) and let U approach 0, we see that the products of operators $R_\alpha(\lambda)^\dagger$ and $\hat{R}_\alpha(\lambda)^\dagger$ act like products of creation operators of Bloch states $c_\alpha^\dagger(k_0)$ on the vacuum. Here $k_0 = p_0$ is determined by the corresponding limit in eq. (4.3),

$$\sin k_0 = -\frac{v}{2}. \tag{6.38}$$

λ and h are now dependent variables. Considering (2.5) and (4.3) for fixed v and small U we find the following solutions

$$\mathrm{i}\cot\lambda = 1 + \frac{U}{4}\left(1 - \frac{v^2}{4}\right)^{-\frac{1}{2}} + O(U^2), \tag{6.39}$$

$$\mathrm{e}^{2h} = \mathrm{i}\left(1 - \frac{v^2}{4}\right)^{\frac{1}{2}} - \frac{v}{2} + O(U^2). \tag{6.40}$$

Using these equations and some standard trigonometric identities we can express all the functions of h and λ, which enter the definition of $\tilde{\mathcal{L}}_m(\lambda)$ (cf. (3.13)) in terms of v and U. Note that (6.39) and (6.40) are not the only possible solution of (2.5) and (4.3) for fixed v and small U. We choose the branches such that $\lim_{U\to 0}\tilde{\mathcal{L}}_m(\lambda)|_v = I_4$. For small U the odd elements of $\tilde{\mathcal{L}}_m(\lambda) - I_4$ are of the order of $U^{\frac{1}{2}}$ and the even elements are of the order of U. Thus only the first sum on the rhs of (3.11) contributes in order $U^{\frac{1}{2}}$ to the odd elements of $\tilde{\mathcal{T}}(\lambda) - I_4$, and we obtain

$$C_{2\alpha}(\lambda) = \mathrm{i}^{\alpha-3}\frac{\mathrm{e}^{-h}}{\cos\lambda}\sum_m c_{m\alpha}^\dagger \mathrm{e}^{-\mathrm{i}mk_0} + O(U^{\frac{3}{2}}), \tag{6.41}$$

$$B_{3-\alpha,1}(\lambda) = \mathrm{i}^{1-\alpha}\frac{\mathrm{e}^{h}}{\cos\lambda}\sum_m c_{m\alpha}^\dagger \mathrm{e}^{-\mathrm{i}mp_0} + O(U^{\frac{3}{2}}), \tag{6.42}$$

where $\mathrm{e}^{\pm h}/\cos\lambda = O(U^{\frac{1}{2}})$. Since $D_{\beta\beta}(\lambda) = 1 + O(U)$ $(\beta = 1, 2)$, it follows from the definitions (6.1) and (6.2) that

$$\lim_{U\to 0} R_\alpha(\lambda)^\dagger|_v = \lim_{U\to 0}\hat{R}_\alpha(\lambda)^\dagger|_v = c_\alpha^\dagger(k_0). \tag{6.43}$$

The corresponding formulae for $R_\alpha(\lambda)$ and $\hat{R}_\alpha(\lambda)$ are true by hermitian conjugation. Eqs. (6.6)-(6.11) turn into the usual anticommutators between fermi operators, since

$$\lim_{U\to 0} r(\lambda,\mu)|_{v,\bar{v}} = \mathcal{P}. \tag{6.44}$$

We see that we may interpret the Zamolodchikov-Faddeev algebra as a deformation of the anticommutators between fermi operators with deformation parameter U.

6.2 Bound States

One of the delicate points in Bethe ansatz calculations is the question of completeness. The completeness of the coordinate Bethe ansatz for the Hubbard model under periodic boundary conditions was discussed by Eßler, Korepin and Schoutens [33]. They showed that the Bethe wave functions are highest weight with respect to the su(2)⊕su(2) symmetry generated by S^a and η^a [33]. Hence, each solution of the Bethe ansatz equations correspond to a su(2)⊕su(2) multiplet. They counted the number of Bethe ansatz solutions assuming Takahashi's string hypothesis [32] to be valid and multiplied by the multiplicities of the su(2)⊕su(2) multiplets. The resulting number is equal to the dimension of the Hilbert space. Note however that, although a matter of common belief now, the string hypothesis is waiting for a proof since 25 years.

How to pose the question of completeness in our infinite chain formalism? As we have seen in the preceding section the operators $R_\alpha(\lambda)^\dagger$ and $\hat{R}_\alpha(\lambda)^\dagger$ create single electrons in scattering states. The operator $D_{21}(\lambda)$ creates a bound pair of electrons. As we have learned in section 5, there are no operators that create more than two particles among the elements of the monodromy matrix. As we will see, however, the string hypothesis suggests the existence of bound states of pairs. How to define the corresponding bound state operators?

To obtain a guess, let us recall the string hypothesis. In the following we will denote the spin rapidities of the coordinate Bethe ansatz by Λ_j and the momenta by k_j (for details cf. [32]). According to the string hypothesis there are two types of string solutions of the Bethe ansatz equations.

1. (Λ-string) m Λ_j's form a string configuration, in which the real parts of the Λ_j's are the same while the imaginary parts are arranged at equal spacing of $\mathrm{i}U/2$. The center of the string should be real.

2. (k-Λ-string) $2m$ κ_i's and m Λ_j's form a string configuration. The values of k_i's and Λ_j's are

$$
\begin{aligned}
k_1 &= \pi - \arcsin(\Lambda' + \mathrm{i}mU/4), \\
k_2 &= \arcsin(\Lambda' + \mathrm{i}(m-2)U/4), \\
k_3 &= \pi - k_2 \\
&\vdots
\end{aligned}
$$

$$
\begin{aligned}
k_{2m-2} &= \arcsin(\Lambda' - \mathrm{i}(m-2)U/4), \\
k_{2m-1} &= \pi - k_{2m-2} \\
k_{2m} &= \pi - \arcsin(\Lambda' - \mathrm{i}mU/4), \\
\Lambda_j &= \Lambda' + \mathrm{i}(m+1-2j)U/4, \ \ \Lambda' \text{ real}, \quad j = 1, 2, \cdots, m.
\end{aligned}
$$

These solutions should be exact in the thermodynamic limit. Since we are dealing with the zero density vacuum, there should be no spin excitations, and we do not have to consider the Λ-string here.

We wish to obtain an operator which creates a $2m$-k-Λ-string (for short we shall simply call it as "$2m$-string"). To begin with, we shall deal with the 2-string ($m = 1$ case). There are 2 particles involved, one with spin up and the other one with spin down. The wave function of the 2-string state is of the form

$$
\sum_{m,n} c_{m\uparrow}^{\dagger} c_{n\downarrow}^{\dagger} \left\{ \theta(m \geq n) \mathrm{e}^{-\mathrm{i}(mk+np)} + \theta(m < n) \mathrm{e}^{-\mathrm{i}(nk+mp)} \right\} |0\rangle, \tag{6.45}
$$

where $\sin k - \sin p = \mathrm{i}U/2$ and $k+p$ real. Therefore, by comparison with eq. (5.6), the 2-string state is proportional to $D_{21}(\lambda)|0\rangle$ with an appropriate choice of λ. It can be easily seen by explicit use of (3.11) and (3.13) that it is also proportional to $C_{22}(\lambda')C_{21}(\lambda'')|0\rangle$, if λ' and λ'' satisfy the following conditions;

$$
p(\lambda') = \pi - k(\lambda'') \mod 2\pi, \tag{6.46}
$$

$$
k(\lambda'') = p(\lambda) \mod 2\pi, \tag{6.47}
$$

$$
k(\lambda') = k(\lambda) \mod 2\pi. \tag{6.48}
$$

These are three conditions for three parameters λ, λ', λ'', which at first sight seems to violate the arbitrariness of λ. Yet there is a redundancy in these equations. (6.46) and (6.47) imply

$$
p(\lambda') = \pi - p(\lambda) \mod 2\pi, \tag{6.49}
$$

which is compatible with (6.48) by taking into account the constraint (5.7). Thus we have obtained two possible 2-string creation operators, which are connected with each other by

$$
\begin{aligned}
D_{21}(\lambda)|0\rangle = & \frac{\mathrm{i}\mathrm{e}^{h(\lambda')+h(\lambda'')} \cos\lambda'' \cos^2\lambda' \sin\lambda'}{\cos^2\lambda} \\
& \cdot \frac{(1 - \mathrm{e}^{\mathrm{i}(p(\lambda')-k(\lambda''))})(1 - \mathrm{e}^{\mathrm{i}(k(\lambda'')-k(\lambda'))})}{1 - \mathrm{e}^{\mathrm{i}(p(\lambda')-k(\lambda'))}} C_{22}(\lambda')C_{21}(\lambda'')|0\rangle.
\end{aligned} \tag{6.50}
$$

λ, λ' and λ'' in this equation have to satisfy (6.46)-(6.48). Note that it follows from (C.5) that

$$C_{22}(\lambda')C_{21}(\lambda'') = -C_{21}(\lambda')C_{22}(\lambda''). \tag{6.51}$$

Let us proceed with the general $2m$-string states. We conjecture that the creation operator of a $2m$-k-Λ-string can be expressed as

$$C_2^{(2m)}(\lambda_1, \cdots, \lambda_{2m}) = C_{22}(\lambda_1)C_{21}(\lambda_2)C_{22}(\lambda_3)C_{21}(\lambda_4)\cdots C_{22}(\lambda_{2m-1})C_{21}(\lambda_{2m}), \tag{6.52}$$

where

$$k(\lambda_{2s}) + p(\lambda_{2s-1}) = \pi \ (\mathrm{mod}\ 2\pi), \tag{6.53}$$

$$\sin k(\lambda_{2s-1}) = \sin k(\lambda_1) + \frac{\mathrm{i}U(s-1)}{2},\ \ (s = 1, \cdots, m). \tag{6.54}$$

Following previous works [56, 57, 58, 59] we shall call this operator bound-state operator. The expression on the rhs is a formal one and should be interpreted as a "composite operator" (see Appendix C). One can easily verify that the functions $\sin k(\lambda_i)$ form the same configuration as in the k-Λ-string, if their center

$$\zeta = \frac{1}{2m}\sum_{i=1}^{2m} \sin k(\lambda_i) = \sin k(\lambda_1) + \frac{\mathrm{i}U(m-2)}{4} \tag{6.55}$$

is real.

We can normalize the bound-state operator by a method similar to that in the case of the scattering states created by $R_\alpha(\lambda)^\dagger$. Let

$$D_{22}^{(2m)}(\lambda_1, \cdots, \lambda_{2m}) = D_{22}(\lambda_1)D_{22}(\lambda_2)D_{22}(\lambda_3)D_{22}(\lambda_4)\cdots D_{22}(\lambda_{2m-1})D_{22}(\lambda_{2m}). \tag{6.56}$$

We define a normalized bound-state operator as

$$R^{(2m)}(\lambda_1, \cdots, \lambda_{2m})^\dagger = C_2^{(2m)}(\lambda_1, \cdots, \lambda_{2m})D_{22}^{(2m)}(\lambda_1, \cdots, \lambda_{2m})^{-1}. \tag{6.57}$$

Similar definitions of bound state operators have appeared before in the literature. They have been applied to the XXZ-chain [58, 59] and to the attractive δ-function gas [56, 60]. Note that it is not a priori clear, if we can apply the commutation rules for the elements of the monodromy matrix to obtain the commutators of $2m$-string operators (cf. Appendix C). However, if we assume we can, we get the following reasonable result,

$$R^{(2m)}(\lambda_i)^\dagger R^{(2n)}(\mu_j)^\dagger = \frac{\zeta - \eta + (n+m)\mathrm{i}U/4}{\zeta - \eta - (n+m)\mathrm{i}U/4}\ \frac{\zeta - \eta + |n-m|\mathrm{i}U/4}{\zeta - \eta - |n-m|\mathrm{i}U/4}$$

$$\cdot \prod_{s=1}^{\min(m,n)-1} \left[\frac{\zeta-\eta+(n+m-2s)\mathrm{i}U/4}{\zeta-\eta-(n+m-2s)\mathrm{i}U/4}\right]^2 R^{(2n)}(\mu_j)^\dagger R^{(2m)}(\lambda_i)^\dagger, \tag{6.58}$$

$$R^{(2m)}(\lambda_i)^\dagger R_a(\mu)^\dagger = \frac{\zeta-\sin k(\mu)+\mathrm{i}Um/4}{\zeta-\sin k(\mu)-\mathrm{i}Um/4} R_a(\mu)^\dagger R^{(2m)}(\lambda_i)^\dagger, \tag{6.59}$$

where ζ is the center of the $2m$-string and η is the center of the $2n$-string. The factor on the rhs of (6.58) is the S-matrix between a $2m$-string and a $2n$-string, and that on the rhs of (6.59) is the S-matrix between a $2m$-string and a particle. (6.58) is of the same form as the S-matrix for the scattering of bound states of magnons in the XXX-chain [59].

As for the transformation under the Yangian Y(su(2)) we can easily show that

$$[Q_0^a, R^{(2m)}(\lambda_i)^\dagger] = 0 = [Q_1^a, R^{(2m)}(\lambda_i)^\dagger], \tag{6.60}$$

which follows from eq. (C.6) and from the commutativity of Q_n^a and $D_{22}(\lambda_j)$. From (6.60) the action of the Yangian Y(su(2)) on a $2m$-string state can be derived as

$$Q_0^a R^{(2m)}(\lambda_i)^\dagger|0\rangle = 0, \quad Q_1^a R^{(2m)}(\lambda_i)^\dagger|0\rangle = 0, \tag{6.61}$$

i.e. the $2m$-string state is singlet under Y(su(2)).

7 Concluding Remarks and Discussion

We have developed the QISM for the Hubbard model on the infinite interval with respect to the zero density vacuum. The R-matrix (3.27) thus obtained is greatly simplified in comparison with the R-matrix of the finite periodic model. In particular, it reveals a hidden rational structure, which arises from a certain combination of the functions ρ_i in eq. (4.2). This structure was discovered earlier by Ramos and Martins [22] as part of the exchange relation for the Hubbard model on the finite interval. Along with the simplified R-matrix we obtained the asymptotic expansion (4.5) of the submatrix $A(\lambda)$ of the monodromy matrix, which naturally provides a representation of Y(su(2)) and generates an infinite series of mutually commuting Yangian invariant operators, which is including the Hamiltonian. We thus clarified the origin of the Yangian symmetry of the Hubbard model.

We constructed creation and annihilation operators of elementary excitations. There are two types of excitations over the zero density vacuum, one-particle excitations and $2m$-string excitations. We showed that they are distinguished by their behavior under the action of

the Yangian. All $2m$-string excitations are Yangian singlets, whereas the n-fold one-particle excitations transform like n-fold tensor products of the fundamental Yangian representation.

The interaction of two elementary excitations is described by their S-matrix, which is given by the commutation relations of the corresponding operators. We calculated these commutation relations and thus the S-matrix. The $2m$-string excitations are Yangian singlet, which means that they have no internal structure. Therefore their mutual interaction produces only phase shifts, their S-matrix (6.58) is a scalar factor. In contrast, the scattering of one-particle excitations involves the spin. The S-matrix is a 4×4-matrix. Up to a factor of minus one, which accounts for the fermionic nature of the excitations, it agrees with the rational submatrix $r(\lambda, \mu)$ of the R-matrix in the thermodynamic limit. We obtained two alternative pairs of one-particle creation operators. The corresponding S-matrices are reciprocal to one-another, which means that, if one of the pairs creates in-states, the other one creates out-states. Our creation and annihilation operators combine into a left and right representation of the Zamolodchikov-Faddeev algebra. Although it should be obvious, we would like to emphasize that our one-particle operators in equation (6.6)-(6.11) are not abstract quantities, but explicitly given in terms of the elementary creation and annihilation operators of Wannier states. In the limit of vanishing coupling ($U \to 0$) our one-particle operators turn into creation and annihilation operators of Bloch states. The Zamolodchikov-Faddeev algebra can thus be understood as a deformation of the anticommutators between fermi operators with deformation parameter U.

There is a known standard method for the solution of the "quantum inverse problem" [27], which consists in expressing the fermi operators for electrons in Wannier states $c_{j\sigma}$, $c_{j\sigma}^\dagger$ in terms of quasi-particle operators $R_\alpha(\lambda)$, $R_\alpha(\lambda)^\dagger$ and $R^{(2m)}(\lambda)$, $R^{(2m)}(\lambda)^\dagger$. In analogy to the classical inverse scattering method, we have to derive a quantum Gelfand-Levitan equation. To this end we should investigate the analytic properties of the monodromy matrix. Due to the complicated structure of the R-matrix, this may be a difficult task. We leave it for future work. The solution of the inverse problem will enable us to calculate Green's function with respect to the considered vacuum, as it was done before for other integrable models [30, 58, 59].

So far our work has been limited to the four cases of uncorrelated vacua, for which the up-spin or down-spin orbitals are either completely filled or vacant. Unfortunately, these cases are not of particular interest from the point of view of condensed matter physics. Correlation functions are rather trivial and may be obtained directly within the coordinate Bethe ansatz

solution of Lieb and Wu. Hence there is a clear need to develop a method for the algebraic construction of excitations over a *correlated* vacuum. In a first step we had to renormalize $\mathcal{T}_{mn}(\lambda)$ with respect to this vacuum. The properties of the vacuum enter the formalism, they are not results of it, and therefore have to be obtained by independent means. Assume we had done this step. Then the next question would be, if the vacuum is a Fock vacuum with respect to some of the entries of the renormalized monodromy matrix. This is, of course, not clear a priori, but is a necessary requirement for an algebraic construction of eigenstates based on commutation relations.

The development of a version of the QISM for interacting vacua is certainly a hard and challenging task, and it may be more appropriate to start with a more simple model than the Hubbard model. It seems however, that such kind of method would finally lead to a complete understanding of the Hubbard model, too. We would like to consider our work as a first necessary step towards this goal.

For a further investigation of the Hubbard model by algebraic means the physically interesting half filled case seems to be most appropriate. At half filling the $2m$-strings disappear from the spectrum [46]. There are only two pairs of *independent* one-particle excitations, which have been described as “spinons” and “holons” [46]. In analogy to the results in section 6.1 we expect the space of states for a given set of rapidities to be spanned by the action of Y(su(2))⊕Y(su(2)) Yangian generators on Yangian highest weight states. We further know that the half filled ground state (at zero magnetic field) is singlet with respect to Y(su(2))⊕Y(su(2)) [42]. This suggests that form factors may be calculated by purely algebraic means [7].

Acknowledgements

We would like to thank M. Wadati for his interest in this work, for his hospitality and for bringing us together. We benefited from stimulating discussions with M. J. Martins, M. Shiroishi, H. Frahm and J. Suzuki. S. M. is grateful to N. Nagaosa for his encouragement. The final preparation of this manuscript during a visit of F. G. at the University of Tokyo became possible through financial support by the DFG (grant number Go 825/1-1). In Bayreuth F. G. is postdoc fellow of the “Graduiertenkolleg nichtlineare Spektroskopie und Dynamik”. He likes to express his gratitude to F. G. Mertens and D. Haarer for their support.

Appendix A Singular Terms in the Infinite-chain Formalism

In this appendix we calculate the limits

$$(U_+(\lambda,\mu)^{-1})_{\alpha\beta,\gamma\delta} = \lim_{n\to\infty}(U_n(\lambda,\mu)^{-1})_{\alpha\beta,\gamma\delta}, \tag{A.1}$$

$$U_-(\lambda,\mu)_{\alpha\beta,\gamma\delta} = \lim_{n\to-\infty}U_n(\lambda,\mu)_{\alpha\beta,\gamma\delta}, \tag{A.2}$$

which determine the elements of $\tilde{\mathcal{T}}(\lambda)\otimes\tilde{\mathcal{T}}(\mu)$ via equation (3.24). $U_n(\lambda,\mu)^{-1}$ and $U_n(\lambda,\mu)$ are defined by eq.(3.22), where $V(\lambda)$ is given according to (3.12). $V^{(2)}(\lambda,\mu)$ is easily obtained by direct calculation. Its diagonal consists of the elements of $V(\lambda)\otimes_{\mathrm{s}} V(\mu)$. Due to normal ordering there appear additional non-vanishing off-diagonal elements,

$$V^{(2)}(\lambda,\mu)_{12,21} = V^{(2)}(\lambda,\mu)_{13,31} = -\mathrm{i}\sin\lambda\sin\mu, \tag{A.3}$$

$$V^{(2)}(\lambda,\mu)_{14,23} = -V^{(2)}(\lambda,\mu)_{14,32} = -\mathrm{i}\sin\lambda\ \cos\mu, \tag{A.4}$$

$$V^{(2)}(\lambda,\mu)_{24,42} = V^{(2)}(\lambda,\mu)_{34,43} = \mathrm{i}\cos\lambda\cos\mu, \tag{A.5}$$

$$V^{(2)}(\lambda,\mu)_{23,41} = -V^{(2)}(\lambda,\mu)_{32,41} = -\mathrm{i}\cos\lambda\sin\mu, \tag{A.6}$$

$$V^{(2)}(\lambda,\mu)_{14,41} = -\mathrm{e}^{h(\lambda)+h(\mu)}. \tag{A.7}$$

Note that $V^{(2)}(\lambda,\mu)$ is upper triangular. Since the diagonals of $V^{(2)}(\lambda,\mu)$ and $V(\lambda)\otimes_{\mathrm{s}} V(\mu)$ agree, $V^{(2)}(\lambda,\mu)$ can be diagonalized by an upper triangular matrix $U(\lambda,\mu)$ whose diagonal elements are all unity, and

$$V^{(2)}(\lambda,\mu) = U(\lambda,\mu)(V(\lambda)\otimes_{\mathrm{s}} V(\mu))U(\lambda,\mu)^{-1}. \tag{A.8}$$

It turns out that the non-vanishing off-diagonal elements of $U(\lambda,\mu)$ are simple rational functions of the Boltzmann weights $\rho_j = \rho_j(\lambda,\mu)$. They are obtained as

$$U(\lambda,\mu)_{12,21} = U(\lambda,\mu)_{13,31} = \frac{-\mathrm{i}\rho_2}{\rho_{10}}, \tag{A.9}$$

$$U(\lambda,\mu)_{14,23} = -U(\lambda,\mu)_{14,32} = \frac{\mathrm{i}\rho_6}{\rho_3-\rho_1}, \tag{A.10}$$

$$U(\lambda,\mu)_{24,42} = U(\lambda,\mu)_{34,43} = \frac{\mathrm{i}\rho_2}{\rho_9}, \tag{A.11}$$

$$U(\lambda,\mu)_{23,41} = -U(\lambda,\mu)_{32,41} = \frac{\mathrm{i}\rho_6}{\rho_5-\rho_4}, \tag{A.12}$$

$$U(\lambda,\mu)_{14,41} = \frac{-\rho_5}{\rho_5-\rho_4}. \tag{A.13}$$

Before proceeding further let us introduce some shorthand notation. Instead of $f(\lambda)$, $g(\mu)$ we will write f, $\bar{g}$. The bar means that the argument of the function if μ. Using this convention

and equation (A.8) we obtain

$$
\begin{aligned}
U_n(\lambda,\mu) &= U(\lambda,\mu)(V^{-n}\otimes_{\mathrm{s}}\bar{V}^{-n})U(\lambda,\mu)^{-1}(V^n\otimes_{\mathrm{s}}\bar{V}^n), &\text{(A.14)}\\
U_n(\lambda,\mu)^{-1} &= (V^{-n}\otimes_{\mathrm{s}}\bar{V}^{-n})U(\lambda,\mu)(V^n\otimes_{\mathrm{s}}\bar{V}^n)U(\lambda,\mu)^{-1}, &\text{(A.15)}
\end{aligned}
$$

Comparing (A.14), (A.15) and (A.1), (A.2) we find that we have to calculate the following limits

$$
\begin{aligned}
l_1^\pm &= \lim_{n\to\pm\infty}\frac{\mathrm{i}\rho_2}{\rho_{10}}(1-\mathrm{e}^{\mathrm{i}n(p-\bar{p})}), &\text{(A.16)}\\
l_2^\pm &= \lim_{n\to\pm\infty}\frac{-\mathrm{i}\rho_2}{\rho_9}(1-\mathrm{e}^{\mathrm{i}n(k-\bar{k})}), &\text{(A.17)}\\
l_3^\pm &= \lim_{n\to\pm\infty}\frac{\mathrm{i}\rho_6}{\rho_1-\rho_3}(1-\mathrm{e}^{\mathrm{i}n(p-\bar{k})}), &\text{(A.18)}\\
l_4^\pm &= \lim_{n\to\pm\infty}\frac{\mathrm{i}\rho_6}{\rho_4-\rho_5}(1-\mathrm{e}^{\mathrm{i}n(k-\bar{p})}), &\text{(A.19)}\\
l_5^+ &= \lim_{n\to\infty}\left\{\frac{\rho_3}{\rho_1-\rho_3}+\frac{2\rho_6^2}{(\rho_1-\rho_3)(\rho_4-\rho_5)}\mathrm{e}^{\mathrm{i}n(p-\bar{k})}\right.\\
&\quad\left.+\frac{\rho_5}{\rho_4-\rho_5}\mathrm{e}^{\mathrm{i}n(k+p-\bar{k}-\bar{p})}\right\}, &\text{(A.20)}\\
l_6^- &= \lim_{n\to-\infty}\left\{\frac{\rho_5}{\rho_4-\rho_5}+\frac{2\rho_6^2}{(\rho_1-\rho_3)(\rho_4-\rho_5)}\mathrm{e}^{\mathrm{i}n(k-\bar{p})}\right.\\
&\quad\left.+\frac{\rho_3}{\rho_1-\rho_3}\mathrm{e}^{\mathrm{i}n(k+p-\bar{k}-\bar{p})}\right\}, &\text{(A.21)}
\end{aligned}
$$

These limits exist in the sense of generalized functions. It turns out that they all can be reduced to the following formula

$$
\lim_{n\to\pm\infty}\frac{1-\mathrm{e}^{\mathrm{i}(p-\bar{p})n}}{\mathrm{e}^{\mathrm{i}\bar{p}}-\mathrm{e}^{\mathrm{i}p}}=\frac{1}{\mathrm{e}^{\mathrm{i}\bar{p}}-\mathrm{e}^{\mathrm{i}(p\pm\mathrm{i}\varepsilon)}}, \tag{A.22}
$$

where ε is an infinitesimal positive number. The limits exist in the domain $\pm\Im(p-\bar{p})\geq 0$ A proof of the above formula can be obtained in two steps. First show by acting on a test function that

$$
\lim_{n\to\pm\infty}\text{p.v.}\frac{\mathrm{e}^{\mathrm{i}(p-\bar{p})n}}{\mathrm{e}^{\mathrm{i}(p-\bar{p})}-1}=\pm\pi\tilde{\delta}(p-\bar{p}), \tag{A.23}
$$

where $\tilde{\delta}$ is the periodic δ-function

$$
\tilde{\delta}(p)=\sum_{n=-\infty}^{\infty}\delta(p-2n\pi). \tag{A.24}
$$

Then use a periodic version of the Plemelj formula (cf. page 501 of ref. [29])

$$
\frac{1}{1-\mathrm{e}^{\mathrm{i}(p\pm\mathrm{i}\varepsilon)}}=\text{p.v.}\frac{1}{1-\mathrm{e}^{\mathrm{i}p}}\pm\pi\tilde{\delta}(p), \tag{A.25}
$$

to obtain (A.22).

Using (A.22) we find, for example,

$$
\begin{aligned}
l_1^{\pm} &= -\frac{\mathrm{i}e^{-h-\bar{h}}}{\sin\lambda\sin\mu}\lim_{n\to\pm\infty}\frac{1-e^{\mathrm{i}(p-\bar{p})n}}{e^{\mathrm{i}\bar{p}}-e^{\mathrm{i}p}} \\
&= -\frac{\mathrm{i}e^{-h-\bar{h}}}{\sin\lambda\sin\mu}\frac{1}{e^{\mathrm{i}\bar{p}}-e^{\mathrm{i}(p\pm\mathrm{i}\varepsilon)}}, \qquad (A.26)
\end{aligned}
$$

and similarly

$$
\begin{aligned}
l_2^{\pm} &= \frac{\mathrm{i}e^{h+\bar{h}}}{\sin\lambda\sin\mu}\frac{1}{e^{\mathrm{i}\bar{k}}-e^{\mathrm{i}(k\pm\mathrm{i}\varepsilon)}}, \qquad (A.27) \\
l_3^{\pm} &= \frac{\mathrm{i}e^{-h+\bar{h}}}{\sin\lambda\sin\mu}\frac{1}{e^{\mathrm{i}\bar{k}}-e^{\mathrm{i}(p\pm\mathrm{i}\varepsilon)}}, \qquad (A.28) \\
l_4^{\pm} &= \frac{\mathrm{i}e^{h-\bar{h}}}{\sin\lambda\sin\mu}\frac{1}{e^{\mathrm{i}\bar{p}}-e^{\mathrm{i}(k\pm\mathrm{i}\varepsilon)}}. \qquad (A.29)
\end{aligned}
$$

Note that the prefactors on the rhs of the above formulae when expressed in terms of k, $\bar{k}$, p, $\bar{p}$ are all regular at the singularities of the second factors. The remaining limits are obtained as

$$
\begin{aligned}
l_5^{+} &= \frac{(1+e^{\mathrm{i}(\bar{k}+\bar{p})})(1+e^{\mathrm{i}(k+p)})}{e^{\mathrm{i}\bar{p}}-e^{\mathrm{i}k}}\left\{\frac{e^{\mathrm{i}\bar{p}}+e^{\mathrm{i}k}}{e^{\mathrm{i}(\bar{k}+\bar{p})}-e^{\mathrm{i}(k+p+\mathrm{i}\varepsilon)}}-\frac{2}{e^{\mathrm{i}\bar{k}}-e^{\mathrm{i}(p+\mathrm{i}\varepsilon)}}\right\}, \qquad (A.30) \\
l_6^{-} &= \frac{(1+e^{\mathrm{i}(\bar{k}+\bar{p})})(1+e^{\mathrm{i}(k+p)})}{e^{\mathrm{i}\bar{k}}-e^{\mathrm{i}p}}\left\{\frac{e^{\mathrm{i}\bar{k}}+e^{\mathrm{i}p}}{e^{\mathrm{i}(\bar{k}+\bar{p})}-e^{\mathrm{i}(k+p-\mathrm{i}\varepsilon)}}-\frac{2}{e^{\mathrm{i}\bar{p}}-e^{\mathrm{i}(k-\mathrm{i}\varepsilon)}}\right\}, \qquad (A.31)
\end{aligned}
$$

The last two equations require consideration of the consistency of the occurring singularities.

In particular, in case of (A.30) the following two conditions must be satisfied, (i) $\Im(p) > \Im(\bar{k})$ if $k+p$ and $\bar{k}+\bar{p}$ are real, (ii) $\Im(k) > \Im(\bar{p})$, if $\bar{k}$ and p are real. These two conditions have to be consistent with the constraint (5.7).

Consider the first condition. We can use equation (5.8) to obtain

$$
\Im(p-\bar{k}) = -\operatorname{arcsinh}\left(\frac{U}{4\cos((k+p)/2)}\right) - \operatorname{arcsinh}\left(\frac{U}{4\cos((\bar{k}+\bar{p})/2)}\right). \qquad (A.32)
$$

Hence $\Im(p-\bar{k})$ is positive for positive U, if $\frac{\pi}{2} < |\frac{k+p}{2}| < \pi$, and positive for negative U, if $|\frac{k+p}{2}| < \frac{\pi}{2}$. Note that the same restrictions on the parameters were obtained in section 5.1 as conditions on $D_{21}(\lambda)$ to create a bound state.

Consider the second condition above. The constraint (5.7) has two branches of solutions for p as a function of k, which are fixed by choosing $\operatorname{sgn}(\Im(p))$. We may therefore choose

$\Im(k) > 0 > \Im(\bar{p})$, and (ii) will be satisfied. Moreover, this choice of branch ensures the prefactor $(\mathrm{e}^{\mathrm{i}\bar{p}} - \mathrm{e}^{\mathrm{i}k})^{-1}$ in (A.30) to be nonsingular as k approaches p on the real axis.

Equation (A.31) may be discussed in a similar way. Now the two conditions become (i) $\Im(k) < \Im(\bar{p})$ if $k+p$ and $\bar{k}+\bar{p}$ are real, and (ii) $\Im(p) > \Im(\bar{k})$, if $\bar{p}$ and k are real. Condition (i) implies the same restriction as above, $\frac{\pi}{2} < |\frac{k+p}{2}| < \pi$ for $U > 0$, and $|\frac{k+p}{2}| < \frac{\pi}{2}$ for $U < 0$. (ii) can again be satisfied by an appropriate choice of branch of p as a function of k.

We are now in a position to consider the weak limits

$$(\tilde{\mathcal{T}}(\lambda) \otimes_s \tilde{\mathcal{T}}(\mu))_{\alpha\beta,\gamma\delta} = (U_+(\lambda,\mu)^{-1})_{\alpha\beta,\epsilon\varphi} \tilde{\mathcal{T}}^{(2)}(\lambda,\mu)_{\epsilon\varphi,\eta\rho} U_-(\lambda,\mu)_{\eta\rho,\gamma\delta}. \tag{A.33}$$

We have to check, whether the functions on the rhs of this equation exist on a common domain. Clearly, this has to be done equation by equation. We find the following 16 combinations of functions $l_j^\pm$:

$$\begin{gathered}
\{l_1^+, l_1^-\},\ \{l_1^+, l_3^-\},\ \{l_3^+, l_5^+, l_1^-\},\ \{l_3^+, l_5^+, l_3^-\},\ \{l_1^+, l_4^-, l_6^-\},\\
\{l_1^+, l_2^-\},\ \{l_3^+, l_5^+, l_4^-, l_6^-\},\ \{l_3^+, l_5^+, l_2^-\},\ \{l_4^+, l_1^-\},\ \{l_4^+, l_3^-\},\\
\{l_2^+, l_1^-\},\ \{l_2^+, l_3^-\},\ \{l_4^+, l_4^-, l_6^-\},\ \{l_4^+, l_2^-\},\ \{l_2^+, l_4^-, l_6^-\},\ \{l_2^+, l_2^-\}.
\end{gathered}$$

It turns out that each of these combinations is compatible. However, they are not compatible all together.

Now we should solve (A.33) for $\tilde{\mathcal{T}}^{(2)}(\lambda,\mu)_{\alpha\beta,\gamma\delta}$ and insert the result into (3.20) to obtain the commutation relations between the elements of the monodromy matrix. We should do that equation by equation and should take care of the compatibility of the domains of the occurring functions. This would be a cumbersome task and would, moreover, obstruct the algebraic structure of our problem. Therefore we leave mathematical rigor at this point and proceed more formally.

First note that there occur products of the form $l_1^+ l_1^-$ on the rhs of (A.33). These products are not well defined, since the regularization requires two limits, and, when acting on a test function, the result will depend on the order of these limits. "This indicates the highly singular operator character"* of some of the elements of the monodromy matrix. In the following we will therefore exclude the singular points of the functions $l_j^\pm$ (A.26) - (A.31). We will assume that $k \neq \bar{k}$ for k, $\bar{k}$ real etc. Then we can omit the regularizations $\pm\mathrm{i}\varepsilon$ in $l_j^\pm$,

*We are citing Sklyanin [28].

and

$$U_+(\lambda,\mu)^{-1} = U(\lambda,\mu)^{-1}, \quad U_-(\lambda,\mu) = U(\lambda,\mu). \tag{A.34}$$

We insert this result into (A.33), treat (A.33) as a matrix equation and solve for $\tilde{\mathcal{T}}^{(2)}(\lambda,\mu)$,

$$\tilde{\mathcal{T}}^{(2)}(\lambda,\mu) = U(\lambda,\mu)(\tilde{\mathcal{T}}(\lambda) \otimes_s \tilde{\mathcal{T}}(\mu))U(\lambda,\mu)^{-1}. \tag{A.35}$$

Now (3.20) implies (3.25), and the R-matrix (3.27) is obtained by using (A.34) in eq.(3.26).

Appendix B List of Commutation Rules

Appendix B.1 Elementary Commutators

In this appendix we provide a complete list of the commutation rules encoded in the exchange relation (3.25) in terms of the submatrices $A(\lambda), \cdots, D(\lambda)$ of the monodromy matrix $\tilde{\mathcal{T}}(\lambda)$. As mentioned in section 3 these submatrices generate sub-algebras of (3.25),

$$r(\lambda,\mu)(A(\lambda) \otimes A(\mu)) = (A(\mu) \otimes A(\lambda))r(\lambda,\mu), \tag{B.1}$$

$$s(\lambda,\mu)(D(\lambda) \otimes D(\mu)) = (D(\mu) \otimes D(\lambda))s(\lambda,\mu), \tag{B.2}$$

$$\frac{\rho_4(\lambda,\mu)}{\rho_1(\lambda,\mu)} r(\lambda,\mu) \begin{pmatrix} B_{1a}(\lambda) \\ B_{2a}(\lambda) \end{pmatrix} \otimes \begin{pmatrix} B_{1a}(\mu) \\ B_{2a}(\mu) \end{pmatrix} = \begin{pmatrix} B_{1a}(\mu) \\ B_{2a}(\mu) \end{pmatrix} \otimes \begin{pmatrix} B_{1a}(\lambda) \\ B_{2a}(\lambda) \end{pmatrix}, \tag{B.3}$$

$$\frac{\rho_4(\lambda,\mu)}{\rho_1(\lambda,\mu) - \rho_3(\lambda,\mu)} r(\lambda,\mu) \begin{pmatrix} B_{11}(\lambda) \\ B_{21}(\lambda) \end{pmatrix} \otimes \begin{pmatrix} B_{12}(\mu) \\ B_{22}(\mu) \end{pmatrix} = \begin{pmatrix} B_{12}(\mu) \\ B_{22}(\mu) \end{pmatrix} \otimes \begin{pmatrix} B_{11}(\lambda) \\ B_{21}(\lambda) \end{pmatrix}, \tag{B.4}$$

$$(C_{a1}(\lambda), C_{a2}(\lambda)) \otimes (C_{a1}(\mu), C_{a2}(\mu)) = (C_{a1}(\mu), C_{a2}(\mu)) \otimes (C_{a1}(\lambda), C_{a2}(\lambda)) \frac{\rho_4(\lambda,\mu)}{\rho_1(\lambda,\mu)} r(\lambda,\mu), \tag{B.5}$$

$$(C_{11}(\lambda), C_{12}(\lambda)) \otimes (C_{21}(\mu), C_{22}(\mu)) = (C_{21}(\mu), C_{22}(\mu)) \otimes (C_{11}(\lambda), C_{12}(\lambda)) \frac{\rho_4(\lambda,\mu)}{\rho_1(\lambda,\mu) - \rho_3(\lambda,\mu)} r(\lambda,\mu). \tag{B.6}$$

The matrix $s(\lambda,\mu)$ in (B.2) is defined as

$$s(\lambda,\mu) = \mathrm{diag}\left(1, \frac{\rho_4}{\rho_4 - \rho_5}, \frac{\rho_1 - \rho_3}{\rho_1}, 1\right) \mathcal{P}. \tag{B.7}$$

The commutators of $B(\lambda)$,$C(\lambda)$ and $D(\lambda)$ with the submatrix $A(\mu)$ are given as

$$
\begin{aligned}
r(\lambda,\mu)\left(\begin{pmatrix}B_{11}(\lambda)\\B_{21}(\lambda)\end{pmatrix}\otimes A(\mu)\right) &= \frac{\mathrm{i}\rho_{10}(\lambda,\mu)}{\rho_4(\lambda,\mu)}\,A(\mu)\otimes\begin{pmatrix}B_{11}(\lambda)\\B_{21}(\lambda)\end{pmatrix}, && \text{(B.8)}\\
r(\lambda,\mu)\left(\begin{pmatrix}B_{12}(\lambda)\\B_{22}(\lambda)\end{pmatrix}\otimes A(\mu)\right) &= -\frac{\mathrm{i}\rho_{1}(\lambda,\mu)}{\rho_9(\lambda,\mu)}\,A(\mu)\otimes\begin{pmatrix}B_{12}(\lambda)\\B_{22}(\lambda)\end{pmatrix}, && \text{(B.9)}\\
\frac{\mathrm{i}\rho_{10}(\lambda,\mu)}{\rho_4(\lambda,\mu)}\,(C_{11}(\lambda),C_{12}(\lambda))\otimes A(\mu) &= (A(\mu)\otimes(C_{11}(\lambda),C_{12}(\lambda))\,r(\lambda,\mu), && \text{(B.10)}\\
-\frac{\mathrm{i}\rho_{1}(\lambda,\mu)}{\rho_9(\lambda,\mu)}\,(C_{21}(\lambda),C_{22}(\lambda))\otimes A(\mu) &= (A(\mu)\otimes(C_{21}(\lambda),C_{22}(\lambda))\,r(\lambda,\mu). && \text{(B.11)}
\end{aligned}
$$

$$
\begin{aligned}
[D_{11}(\lambda),A(\mu)] &= [D_{22}(\lambda),A(\mu)] \;=\; 0, && \text{(B.12)}\\
D_{12}(\lambda)A(\mu) &= -\frac{\rho_1(\lambda,\mu)}{\rho_9(\lambda,\mu)}\frac{\rho_4(\lambda,\mu)}{\rho_{10}(\lambda,\mu)}\,A(\mu)D_{12}(\lambda), && \text{(B.13)}\\
D_{21}(\lambda)A(\mu) &= -\frac{\rho_9(\lambda,\mu)}{\rho_1(\lambda,\mu)}\frac{\rho_{10}(\lambda,\mu)}{\rho_4(\lambda,\mu)}\,A(\mu)D_{21}(\lambda). && \text{(B.14)}
\end{aligned}
$$

The commutators between the entries of $B(\lambda)$ and $C(\mu)$ are

$$
\begin{aligned}
B_{a1}(\lambda)C_{1b}(\mu) &= -\frac{\rho_{10}(\lambda,\mu)^2}{\rho_1(\lambda,\mu)\rho_4(\lambda,\mu)}C_{1b}(\mu)B_{a1}(\lambda), && \text{(B.15)}\\
B_{a1}(\lambda)C_{2b}(\mu) &= \frac{\rho_{10}(\lambda,\mu)}{\rho_9(\lambda,\mu)}C_{2b}(\mu)B_{a1}(\lambda), && \text{(B.16)}\\
B_{a2}(\lambda)C_{1b}(\mu) &= \frac{\rho_{10}(\lambda,\mu)}{\rho_9(\lambda,\mu)}C_{1b}(\mu)B_{a2}(\lambda), && \text{(B.17)}\\
B_{a2}(\lambda)C_{2b}(\mu) &= -\frac{\rho_1(\lambda,\mu)\rho_4(\lambda,\mu)}{\rho_9(\lambda,\mu)^2}C_{2b}(\mu)B_{a2}(\lambda). && \text{(B.18)}
\end{aligned}
$$

Finally, there are the following commutators between $B(\lambda)$, $C(\lambda)$ and the submatrix $D(\lambda)$,

$$
\begin{aligned}
B_{a1}(\lambda)D(\mu) &= \mathrm{i}\begin{pmatrix}\frac{\rho_{10}}{\rho_4} & 0\\ 0 & -\frac{\rho_1}{\rho_9}\end{pmatrix}D(\mu)\begin{pmatrix}1 & 0\\ 0 & \frac{\rho_1-\rho_3}{\rho_1}\end{pmatrix}B_{a1}(\lambda), && \text{(B.19)}\\
B_{a2}(\lambda)D(\mu) &= \mathrm{i}\begin{pmatrix}\frac{\rho_{10}}{\rho_4} & 0\\ 0 & -\frac{\rho_1}{\rho_9}\end{pmatrix}D(\mu)\begin{pmatrix}\frac{\rho_4}{\rho_4-\rho_5} & 0\\ 0 & 1\end{pmatrix}B_{a2}(\lambda), && \text{(B.20)}\\
C_{1a}(\lambda)D(\mu) &= -\mathrm{i}\begin{pmatrix}1 & 0\\ 0 & \frac{\rho_1}{\rho_1-\rho_3}\end{pmatrix}D(\mu)\begin{pmatrix}\frac{\rho_4}{\rho_{10}} & 0\\ 0 & -\frac{\rho_9}{\rho_1}\end{pmatrix}C_{1a}(\lambda), && \text{(B.21)}\\
C_{2a}(\lambda)D(\mu) &= -\mathrm{i}\begin{pmatrix}\frac{\rho_4-\rho_5}{\rho_4} & 0\\ 0 & 1\end{pmatrix}D(\mu)\begin{pmatrix}\frac{\rho_4}{\rho_{10}} & 0\\ 0 & -\frac{\rho_9}{\rho_1}\end{pmatrix}C_{2a}(\lambda). && \text{(B.22)}
\end{aligned}
$$

Appendix B.2 Commutators with the quantum determinant

Eqs. (B.8)-(B.14) imply the following commutators of the quantum determinant $\mathrm{Det}_q(A(\mu))$ with the remaining entries of the monodromy matrix

$$\mathrm{Det}_q(A(\mu))B_{a1}(\lambda) = -\frac{\rho_4(\lambda,\mu)\rho_4(\lambda,\check{\mu})}{\rho_{10}(\lambda,\mu)\rho_{10}(\lambda,\check{\mu})}\frac{v(\lambda)-v(\mu)}{v(\lambda)-v(\mu)+\mathrm{i}U}B_{a1}(\lambda)\mathrm{Det}_q(A(\mu)), \quad \text{(B.23)}$$

$$\mathrm{Det}_q(A(\mu))B_{a2}(\lambda) = -\frac{\rho_9(\lambda,\mu)\rho_9(\lambda,\check{\mu})}{\rho_1(\lambda,\mu)\rho_1(\lambda,\check{\mu})}\frac{v(\lambda)-v(\mu)}{v(\lambda)-v(\mu)+\mathrm{i}U}B_{a2}(\lambda)\mathrm{Det}_q(A(\mu)), \quad \text{(B.24)}$$

$$\mathrm{Det}_q(A(\mu))C_{1a}(\lambda) = -\frac{\rho_{10}(\lambda,\mu)\rho_{10}(\lambda,\check{\mu})}{\rho_4(\lambda,\mu)\rho_4(\lambda,\check{\mu})}\frac{v(\lambda)-v(\mu)+\mathrm{i}U}{v(\lambda)-v(\mu)}C_{1a}(\lambda)\mathrm{Det}_q(A(\mu)), \quad \text{(B.25)}$$

$$\mathrm{Det}_q(A(\mu))C_{2a}(\lambda) = -\frac{\rho_1(\lambda,\mu)\rho_1(\lambda,\check{\mu})}{\rho_9(\lambda,\mu)\rho_9(\lambda,\check{\mu})}\frac{v(\lambda)-v(\mu)+\mathrm{i}U}{v(\lambda)-v(\mu)}C_{2a}(\lambda)\mathrm{Det}_q(A(\mu)). \quad \text{(B.26)}$$

$$\mathrm{Det}_q(A(\mu))D_{12}(\lambda) = \frac{\rho_9(\lambda,\mu)\rho_9(\lambda,\check{\mu})}{\rho_1(\lambda,\mu)\rho_1(\lambda,\check{\mu})}\frac{\rho_{10}(\lambda,\mu)\rho_{10}(\lambda,\check{\mu})}{\rho_4(\lambda,\mu)\rho_4(\lambda,\check{\mu})}D_{12}(\lambda)\mathrm{Det}_q(A(\mu)), \quad \text{(B.27)}$$

$$\mathrm{Det}_q(A(\mu))D_{21}(\lambda) = \frac{\rho_1(\lambda,\mu)\rho_1(\lambda,\check{\mu})}{\rho_9(\lambda,\mu)\rho_9(\lambda,\check{\mu})}\frac{\rho_4(\lambda,\mu)\rho_4(\lambda,\check{\mu})}{\rho_{10}(\lambda,\mu)\rho_{10}(\lambda,\check{\mu})}D_{21}(\lambda)\mathrm{Det}_q(A(\mu)), \quad \text{(B.28)}$$

$$[\mathrm{Det}_q(A(\mu)), D_{11}(\lambda)] = 0 = [\mathrm{Det}_q(A(\mu)), D_{22}(\lambda)], \quad \text{(B.29)}$$

where $\check{\mu}$ is defined by $v(\check{\mu}) = v(\mu) - \mathrm{i}U$.

Appendix C Composite operators

For the construction of bound state operators corresponding to the $2m$-string in section 6.2, we have to introduce composite operators, which are formal products of entries of the monodromy matrix. Our definition of the 2-string creation operator, for instance, is

$$R^{(2)}(\lambda_1,\lambda_2)^\dagger = C_{22}(\lambda_1)C_{21}(\lambda_2)D_{22}(\lambda_2)^{-1}D_{22}(\lambda_1)^{-1}, \quad \text{(C.1)}$$

where

$$p(\lambda_1) + k(\lambda_2) = \pi \ (\mathrm{mod}\ 2\pi). \quad \text{(C.2)}$$

Because of the constraint (5.7), this implies

$$\sin k(\lambda_1) - \sin k(\lambda_2) = \frac{\mathrm{i}U}{2}. \quad \text{(C.3)}$$

Thus k_1 and k_2 cannot both be real, and it follows from (5.4) and (5.5) that $C_{22}(\lambda_1)$ and $C_{21}(\lambda_2)$ in the products on the rhs of (C.1) cannot both create bounded states. They are not simultaneously physical operators. Despite this fact, we can provide *the product* $C_{22}(\lambda_1)C_{21}(\lambda_2)$ with physical meaning, when $\sin k(\lambda_1) - \mathrm{i}U/4$ is real. Eq. (A.33) implies

$$C_{22}(\lambda_1)C_{21}(\lambda_2) = l_3^- \tilde{\mathcal{T}}^{(2)}(\lambda_1,\lambda_2)_{44,14} + \tilde{\mathcal{T}}^{(2)}(\lambda_1,\lambda_2)_{44,32}. \tag{C.4}$$

The matrix elements $\tilde{\mathcal{T}}^{(2)}(\lambda_1,\lambda_2)_{\alpha\beta,\gamma\delta}$ are well-defined through a series representation similar to (3.11). We may take (C.4) as a definition of the composite operator on the left hand side. Its domain is the domain of l_3^-. It is determined by the condition $\Im(p(\lambda_1)) \le \Im(k(\lambda_2))$, which leads to the same discussion as above eq. (5.8) or below eq. (A.31).

In principle, we could iterate the renormalization procedure explained in section 3 and in Appendix A. We could define $\mathcal{L}_m^{(k)}(\lambda_1,\cdots,\lambda_m)$ as the k-fold graded tensor product of L-matrices at site m and could introduce its expectation value $V^{(k)}(\lambda_1,\cdots,\lambda_m)$ which governs the renormalization of the k-fold tensor product of monodromy matrices $\mathcal{T}_{mn}^{(k)}(\lambda_1,\cdots,\lambda_k)$. We would obtain a renormalized tensor product $\tilde{\mathcal{T}}^{(k)}(\lambda_1,\cdots,\lambda_k)$ and the commutation relations between the entries of $\tilde{\mathcal{T}}^{(k)}(\lambda_1,\cdots,\lambda_k)$ and $\tilde{\mathcal{T}}^{(l)}(\mu_1,\cdots,\mu_l)$ (let $\tilde{\mathcal{T}}^{(1)}(\lambda) = \tilde{\mathcal{T}}(\lambda)$). We guess that such kind of procedure would solve the completeness problem in a satisfactory way. Unfortunately, it seems to be practically impossible to do these calculation, because of the increasing dimension of the involved matrices.

A composite operator is not a mere product of the original operators. So it is not obvious whether or not the commutation rules between composite operators follow from iterating the commutation rules of its factors, as obtained from (3.25). Nevertheless, we assumed so and investigated some of the consequences of this assumption. This way we obtained the S-matrices (6.58) and (6.59) for composite operators, which look very reasonable. Another consequence of such kind of formal procedure is that all $2m$-string states are Yangian singlet. In case of the two-string this can be seen as follows. Using (B.5) with $\rho_4(\lambda_2,\lambda_1) \neq 0$, $\rho_1(\lambda_2,\lambda_1) = 0$ and $r(\lambda_2,\lambda_1) = (1+\mathcal{P})/2$, we get

$$C_{2\alpha}(\lambda_1)C_{2\beta}(\lambda_2) = -C_{2\beta}(\lambda_1)C_{2\alpha}(\lambda_2). \tag{C.5}$$

From this equation one can derive the following formula

$$[Q_0^a, C_{22}(\lambda_1)C_{21}(\lambda_2)] = 0 = [Q_1^a, C_{22}(\lambda_1)C_{21}(\lambda_2)], \tag{C.6}$$

which indicates that the composite operator $C_{22}(\lambda_1)C_{21}(\lambda_2)$ creates a Yangian invariant pair of up- and down-spin particles.

Appendix D Conserved Quantities of the Hubbard Model

Explicit expressions for higher conserved quantities have been derived by several authors [44, 45]. In contrast to the case of the nonlinear Schrödinger model these quantities are not in one to one correspondence to the conserved quantities of the free fermion model. Let us define creation operators of Bloch states as

$$c_\sigma^\dagger(k) = \sum_j c_{j\sigma}^\dagger \mathrm{e}^{-\mathrm{i}jk}, \quad k \in [-\pi, \pi]. \tag{D.1}$$

In terms of these operators we can define two sequences of operators

$$F_n = -\frac{1}{\pi}\int_{-\pi}^{\pi} dk\ \sin\left(n\left(k+\frac{\pi}{2}\right)\right) c_\sigma^\dagger(k) c_\sigma(k), \tag{D.2}$$

$$\tilde{F}_n = -\frac{1}{\pi}\int_{-\pi}^{\pi} dk\ \cos\left(n\left(k+\frac{\pi}{2}\right)\right) c_\sigma^\dagger(k) c_\sigma(k). \tag{D.3}$$

All of these operators are mutually commuting, and we have $\hat{H}(U=0) = F_1$. Using the results of ref. [45] we obtain the following higher conserved quantities.

$$H_1 = \hat{H}, \tag{D.4}$$

$$\begin{aligned} H_2 = &-\mathrm{i}\sum_{j,\sigma}(c_{j+2,\sigma}^\dagger c_{j,\sigma} - c_{j,\sigma}^\dagger c_{j+2,\sigma}) \\ &+\mathrm{i}U\sum_{j,\sigma}[(c_{j+1,\sigma}^\dagger c_{j,\sigma} - c_{j,\sigma}^\dagger c_{j+1,\sigma})(n_{j,-\sigma}+n_{j+1,-\sigma}-1)], \end{aligned} \tag{D.5}$$

$$\begin{aligned} H_3 = &\sum_{j,\sigma}(c_{j+3,\sigma}^\dagger c_{j,\sigma} + c_{j,\sigma}^\dagger c_{j+3,\sigma}) \\ &- U\sum_{j,\sigma}\Big[(c_{j+2,\sigma}^\dagger c_{j,\sigma} + c_{j,\sigma}^\dagger c_{j+2,\sigma})(n_{j,-\sigma}+n_{j+1,-\sigma}+n_{j+2,-\sigma}-\frac{3}{2}) \\ &\quad + (c_{j+2,\sigma}^\dagger c_{j+1,\sigma} - c_{j+1,\sigma}^\dagger c_{j+2,\sigma})(c_{j+1,-\sigma}^\dagger c_{j,-\sigma} - c_{j,-\sigma}^\dagger c_{j+1,-\sigma}) \\ &\quad + \frac{1}{2}(c_{j+1,\sigma}^\dagger c_{j,\sigma} - c_{j,\sigma}^\dagger c_{j+1,\sigma})(c_{j+1,-\sigma}^\dagger c_{j,-\sigma} - c_{j,\sigma}^\dagger c_{j+1,-\sigma}) \\ &\quad - \left(n_{j+1,\sigma}-\frac{1}{2}\right)\left(n_{j,-\sigma}-\frac{1}{2}\right) - \frac{1}{2}\left(n_{j,\sigma}-\frac{1}{2}\right)\left(n_{j,-\sigma}-\frac{1}{2}\right)+\frac{3}{8}\Big] \\ &\quad + U^2\sum_{j,\sigma}(c_{j+1,\sigma}^\dagger c_{j,\sigma} + c_{j,\sigma}^\dagger c_{j+1,\sigma})\left[\left(n_{j+1,-\sigma}-\frac{1}{2}\right)\left(n_{j,-\sigma}-\frac{1}{2}\right)+\frac{1}{4}\right], \end{aligned} \tag{D.6}$$

$$\begin{aligned} H_4 = &\ \mathrm{i}\sum_{j,\sigma}(c_{j+4,\sigma}^\dagger c_{j,\sigma} - c_{j,\sigma}^\dagger c_{j+4,\sigma}) \\ &- \mathrm{i}U\sum_{j,\sigma}[(c_{j+3,\sigma}^\dagger c_{j,\sigma} - c_{j,\sigma}^\dagger c_{j+3,\sigma})(n_{j,-\sigma}+n_{j+1,-\sigma}+n_{j+2,-\sigma}+n_{j+3,-\sigma}-2) \\ &\quad + (c_{j+2,\sigma}^\dagger c_{j,\sigma} + c_{j,\sigma}^\dagger c_{j+2,\sigma}) \end{aligned}$$

$$
\begin{aligned}
&\cdot (c^\dagger_{j,-\sigma}c_{j-1,-\sigma} - c^\dagger_{j-1,-\sigma}c_{j,-\sigma} + c^\dagger_{j+1,-\sigma}c_{j,-\sigma} - c^\dagger_{j,-\sigma}c_{j+1,-\sigma} + \\
&c^\dagger_{j+2,-\sigma}c_{j+1,-\sigma} - c^\dagger_{j+1,-\sigma}c_{j+2,-\sigma} + c^\dagger_{j+3,-\sigma}c_{j+2,-\sigma} - c^\dagger_{j+2,-\sigma}c_{j+3,-\sigma}) \\
&- (c^\dagger_{j+1,\sigma}c_{j,\sigma} - c^\dagger_{j,\sigma}c_{j+1,\sigma})(n_{j-1,-\sigma} + n_{j,-\sigma} + n_{j+1,-\sigma} + n_{j+2,-\sigma} - 2)] \\
&+ \mathrm{i}U^2 \sum_{j,\sigma} \Big\{ \ (c^\dagger_{j+2,\sigma}c_{j,\sigma} - c^\dagger_{j,\sigma}c_{j+2,\sigma}) \left[\left(n_{j+1,-\sigma} - \frac{1}{2} \right) \left(n_{j,-\sigma} - \frac{1}{2} \right) \right. \\
&\quad \left. + \left(n_{j+2,-\sigma} - \frac{1}{2} \right) \left(n_{j,-\sigma} - \frac{1}{2} \right) + \left(n_{j+2,-\sigma} - \frac{1}{2} \right) \left(n_{j+1,-\sigma} - \frac{1}{2} \right) \right] \\
&+ (c^\dagger_{j+1,\sigma}c_{j,\sigma} + c^\dagger_{j,\sigma}c_{j+1,\sigma}) \left[(c^\dagger_{j,-\sigma}c_{j-1,-\sigma} + c^\dagger_{j-1,-\sigma}c_{j,-\sigma}) \left(n_{j+1,-\sigma} - \frac{1}{2} \right) + \right. \\
&\quad \left. + (c^\dagger_{j+2,-\sigma}c_{j+1,-\sigma} + c^\dagger_{j+1,-\sigma}c_{j+2,-\sigma}) \left(n_{j,-\sigma} - \frac{1}{2} \right) \right] \Big\} \\
&+ \mathrm{i}U^3 \sum_{j,\sigma} \frac{1}{4} (c^\dagger_{j+1,\sigma}c_{j,\sigma} - c^\dagger_{j,\sigma}c_{j+1,\sigma})(n_{j,-\sigma} + n_{j+1,-\sigma} - 1). \qquad \text{(D.7)}
\end{aligned}
$$

These operators have the following properties, (i) they are Hermitian and renormalized, $H_s|0\rangle = 0$, (ii) they are SO(4) invariant, $[H_s, S^a] = 0 = [H_s, \eta^a]$, (iii) $H_s(U = 0) = F_s$, (iv) their action on the one-particle states $|k, \sigma\rangle = c^\dagger_\sigma(k)|0\rangle$ is obtained as

$$
\begin{aligned}
H_1|k,\sigma\rangle &= \left[-2\cos k - \frac{U}{2} \right] |k,\sigma\rangle, && \text{(D.8)} \\
H_2|k,\sigma\rangle &= [2\sin 2k + 2U \sin k] \, |k,\sigma\rangle, && \text{(D.9)} \\
H_3|k,\sigma\rangle &= \left[2\cos 3k - \frac{3U}{2} + 3U\cos 2k + U^2 \cos k \right] |k,\sigma\rangle, && \text{(D.10)} \\
H_4|k,\sigma\rangle &= \left[-2\sin 4k + U(4\sin k - 4\sin 3k) - \frac{3U^2}{2}\sin 2k + \frac{U^3}{2}\sin k \right] |k,\sigma\rangle. && \text{(D.11)}
\end{aligned}
$$

The latter formulae lead us to the conjecture (5.24) in section 5.3. We make a remark here that there is an arbitrariness in the definition of higher conserved quantities by adding linear combinations of lower ones. We do not know what is the most natural choice for higher conserved quantities.

References

[1] L. A. Takhtajan and L. D. Faddeev, Russ. Math. Surv. 34 (1979) 11.

[2] P. P. Kulish and E. K. Sklyanin, J. Sov. Math. 19 (1982) 1596.

[3] P. P. Kulish and E. K. Sklyanin, Lecture Notes in Physics 151 (Springer Verlag, Berlin-New York, 1982)p. 61.

[4] M. Jimbo, ed., *Yang-Baxter equation in integrable systems*, (World Scientific, Singapore, 1990).

[5] V. E. Korepin, N. M. Bogoliubov and A. G. Izergin, *Quantum Inverse Scattering Method and Correlation Functions* (Cambridge University Press, Cambridge, 1993).

[6] F. A. Smirnov, *Form Factors in Completely Integrable Models of Quantum Field Theory* (World Scientific, Singapore, 1992).

[7] M. Jimbo and T. Miwa, *Algebraic Analysis of Solvable Lattice Models*, (American Mathematical Society, 1995).

[8] V. E. Korepin and F. H. L. Eßler, eds, *Exactly Solvable Models of Strongly Correlated Electrons* (World Scientific, Singapore, 1994).

[9] A. M. Tsvelick and P. B. Wiegmann, Adv. Phys. 32 (1983) 453.

[10] N. Andrei, K. Furuya and J. H. Lowenstein, Rev. Mod. Phys. 55 (1983) 331.

[11] E. H. Lieb and F. Y. Wu, Phys. Rev. Lett. 20 (1968) 1445.

[12] C. N. Yang, Phys. Rev. Lett. 19 (1967) 1312.

[13] M. Gaudin, Phys. Lett. A24 (1967) 55.

[14] B. S. Shastry, Phys. Rev. Lett. 56 (1986) 1529

[15] B. S. Shastry, Phys. Rev. Lett. 56 (1986) 2453.

[16] B. S. Shastry, J. Stat. Phys. 50 (1988) 57.

[17] M. Wadati, E. Olmedilla and Y. Akutsu, J. Phys. Soc. Jpn. 56 (1987) 1340.

[18] E. Olmedilla, M. Wadati and Y. Akutsu, J. Phys. Soc. Jpn. 56 (1987) 2298.

[19] E. Olmedilla and M. Wadati, Phys. Rev. Lett. 60 (1988) 1595.

[20] V. O. Tarasov, Theor. Math. Phys. 76 (1988) 793.

[21] A. G. Izergin and V. E. Korepin, Commun. Math. Phys. 79 (1981) 303.

[22] P. B. Ramos and M. J. Martins, J. Phys. A: Math. Gen. A30 (1997) L195.

[23] S. Murakami and F. Göhmann, Phys. Lett. A227 (1997) 216.

[24] S. Murakami and M. Wadati, J. Phys. Soc. Jpn. 65 (1996) 1227.

[25] S. Murakami and M. Wadati, J. Phys. A: Math. Gen. 29 (1996) 7903.

[26] L. D. Faddeev, Sov. Sci. Rev. Math. Phys. C1 (1980) 107.

[27] H. B. Thacker, Rev. Mod. Phys. 53 (1981) 253.

[28] E. K. Sklyanin, J. Sov. Math. 19 (1982) 1546.

[29] L. D. Faddeev and L. A. Takhtajan, *Hamiltonian Methods in the Theory of Solitons* (Springer-Verlag, Berlin, 1987).

[30] D. B. Creamer, H. B. Thacker and D. Wilkinson, Phys. Rev. D21 (1980)1523.

[31] F. Göhmann and S. Murakami, J. Phys. A: Math. Gen. 30 (1997) 5269.

[32] M. Takahashi, Prog. Theor. Phys. 47 (1972) 69.

[33] F. H. L. Eßler, V. E. Korepin and K. Schoutens, Phys. Rev. Lett. 67 (1991) 3848; Nucl. Phys. B372 (1992) 559; *ibid.* B384 (1992) 431.

[34] O. J. Heilmann and E. H. Lieb, Ann. NY Acad. Sci. 172 (1971) 583.

[35] C. N. Yang, Phys. Rev. Lett. 63 (1989)2144; C. N. Yang and S. C. Zhang, Mod. Phys. Lett. B4 (1990) 759.

[36] R. Yue and T. Deguchi, J. Phys. A: Math. Gen. 30 (1997) 849.

[37] F. D. M. Haldane, in Proc. of the 16th Taniguchi Symposium on Condensed Matter Physics (Springer, Berlin-Heidelberg-New York, 1994).

[38] D. Bernard, M. Gaudin, F. D. M. Haldane and V. Pasquier, J. Phys. A26 (1993) 5219.

[39] V. G. Drinfeld, in Proc. of the 1986 International Congress of Mathematics (American Mathematical Society, Berkeley, 1986).

[40] V. Chari and A. N. Pressley, L'Enseignement Mathématique 36 (1990) 267; J. Reine. Angew. Math. 417 (1991) 87.

[41] V. Chari and A. N. Pressley, *A Guide to Quantum Groups* (Cambridge University Press, Cambridge, 1994).

[42] D. B. Uglov and V. E. Korepin, Phys. Lett. A190 (1994) 238.

[43] F. Göhmann and V. Inozemtsev, Phys. Lett. A214 (1996) 161.

[44] H. Grosse, Lett. Math. Phys. 18 (1989) 151.

[45] M. P. Grabowski and P. Mathieu, Ann. Phys. 243 (1995) 299.

[46] F. H. L. Eßler and V. E. Korepin, Phys. Rev. Lett. 72 (1994) 908; Nucl. Phys. B426 (1994) 505.

[47] M. Shiroishi, H. Ujino and M. Wadati, preprint cond-mat/9708064 (1997).

[48] A. Polychronakos, Phys. Rev. Lett. 69 (1992) 703; J. A. Minahan and A. Polychronakos, Phys. Lett. B302 (1993) 265.

[49] F. C. Pu, Y. Z. Wu and B. H. Zhao, J. Phys. A: Math. Gen. 20 (1987) 1173.

[50] A. B. Zamolodchikov and Al. B. Zamolodchikov, Ann. Phys. 120 (1979) 253.

[51] I. M. Khamitov, Theor. Math. Phys. 62 (1985) 217: *ibid.* 63 (1985) 486; J. Sov. Math. 40 (1988) 115.

[52] C. Gómez, M. Ruiz-Altaba and G. Sierra, *Quantum Groups in Two-Dimensional Physics* (Cambridge University Press, Cambridge, 1996).

[53] V. E. Korepin, Theor. Math. Phys. 76 (1980) 165.

[54] N. Andrei, J. H. Lowenstein, Phys. Lett. B91 (1980) 401.

[55] A. LeClair and F. A. Smirnov, Int. J. Mod. Phys. A7 (1992) 2997.

[56] M. Göckeler, Z. Phys. C–Particles and Fields 7 (1981) 263; *ibid.*11 (1981) 125; Y. K. Zhou, Z. Phys. C–Particles and Fields 39 (1988) 215.

[57] K. Sogo and M. Wadati, Prog. Theor. Phys. 69 (1983) 431.

[58] P. P. Kulish and E. K. Sklyanin, Phys. Lett. A70 (1979) 461.

[59] P. P. Kulish and F. A. Smirnov, Phys. Lett. A90 (1982) 74; J. Phys. C18 (1985) 1037.

[60] Y. K. Zhou, Z. Phys. C-Particles and Fields, 39 (1988) 215; F. C. Pu and B. H. Zhao, Nucl. Phys. B275 (1986) 77.